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Common Fixed Points for Weak and Strong Convergence Results

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ABSTRACT

In this paper, we study the approximation of common fixed points for more general classes of mappings through weak and strong convergence results of an iterative scheme in a uniformly convex Banach space. Our results extend and improve some known recent results.

1. INTRODUCTION

Let D be a nonempty subset of Banach space X. A mapping $T : D \to D$ is called asymptotically nonexpansive if for a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1, \|T^n x - T^n y\| \le k_n \|x - y\|$ holds for all $x, y \in D$ and $n = 1,2...T$ is also called uniform $(L-\alpha)-$ Lipschitz if for some $a > 0, L > 0, \|T^n x - T^n y\| \le L\|x - y\|_a$ for all $x, y \in D$ and $n = 1,2...$ Moreover, T is termed as nonexpansive if $\|Tx - y\| \le \|x - y\|$ for all $x, y \in D$ and quasi-nonexpansive if $F(T) \ne \theta$ and $\|Tx - y\| \le \|x - y\|$ for all $x \in D$ and $y \in f(T)$. The mapping T is called asymptotically quasi-nonepansive if $F(T) \ne \theta$ and there exists a sequence $\{k_n\}$ in $[1, \infty)$ with $\lim_{n\to\infty} k_n - 1,$ such that $\|T^n x - y\| \le k_n \|x - y\|$ for all $x \in D, y \in F(T)$ and $n = 1,2^{n\to\infty}....$

Das and Debata [1] considered the following iteration scheme for two quasi-nonexpansive mappings S and T as follows:

$$\begin{cases} x_1 \in D \\ x_{n+1} = (1 - a_n)x_n + a_n S y_n \\ y_n = (1 - \beta_n)x_n + \beta_n T x_n, \; for\, all\, n - 1,2... \end{cases}$$

where andare in Takahashi and Tamura [4] studied the above scheme for two nanexpansive mappings.

Recently, Khan and Takahashi [2] studied the above scheme for two asymptotically nonexpansive mappings S and T through weak and strong convergence of the sequence defined by:

$$[x1 \in D$$

$$x_{n+1} - (1 - a_n)x_n + a_n S^n [(1 - \beta_n)x_n + \beta_n T_n x_n]$$

for all n=1,2… $\{a_n\}$ where $\{\beta_n\}$ and in [0,1].

Key words and phrases: Asymptotically quasi-nonexpansive mapping iterative scheme weak and strong convergence, Opial’s condition, Uniform convex Banach space, Uniform (L-a)-Lipschitz, Common fixed point and continuous mappings.

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On the other hand Bose and Laskar [5] studied the following existence theorem:

Theorem 1.1. Let X be a uniformly convex Banach space and D be a nonempty closed convex bounded subset of X and let $S,T:D\leftarrow D$ be a continuous mappings such that for each $x,y\in D$ and n=1,2,…

$$\left\|S^n x-T^n y\right\|\leq a_n\left\|x-y\right\|+\left(\left\|x-S^n x\right\|+\left\|y-T^n y\right\|\right)$$

(1.1) $$+c_n\left(\left\|x-T^n y\right\|+\left\|y-S^n x\right\|\right)$$

where $a_n,b_n,c_n\geq 0$ and satisfying following conditions:

(i) there is an integer I such that $b_n+c_n<1\forall n=1,2...$

(ii) $\lim_{n\to\infty}\frac{a_n+3b_n+c_n}{1-b_n-c_n}=1,$

(iii) $a_n+2c_n<1,$ for at least one n.

Then S and T have unique common fired points and it is unique as fixed points of each S and T.

If we put $b_n=0$ then the condition (1.1) reduces to

(1.2) $$\left\|S^n x-T^n y\right\|\leq a_n\left\|x-y\right\|+c_n\left(\left\|x-T^n y\right\|+\left\|y-S^n x\right\|\right)$$

for all $x,y\in D$ and $n=1,2...$, where $a_n,c_n\geq 0$ with $c_n<1$ and $\lim_{n\to\infty}\frac{a_n+c_n}{1-c_n}=1.$

In this paper, we study the problems of approximation of common fixed points for uniform Lipschitz asymptotically quasi-nonexpansive mappings and also for the continuous mappings which satisfy the condition (1.2). Our scheme is given by the sequence $\{x_n\}$ in D defined as follows:

(1.3) $$\begin{cases} x_1\in D \\ x_{n+1}=(1-a_n)x_n+a_nT^n y_n \\ y_n=(1-\beta_n)x_n+\beta_n S^n x_n, n=1,2... \end{cases}$$

Where $\{\alpha_n\}$ and β_n are sequences in [0, 1]. Our results improve and extend the corresponding previously known results of khan and Takahashi [3].

2. PERLIMINARIES

We give the following Lemmas which we shall need in the sequel.

Lemma 2.1. [8] Let $\{r_n\},\{s_n\},\{t_n\}$ be three nonnegative sequence satisfying the following condition.

$$r_{n+1}\leq(1+s_n)r_n+t_n \ for\, all\, n\in N.$$

If $\sum_{n=1}^{\infty}s_n<\infty,\sum_{n=1}^{\infty}t_n<\infty$. Then $\lim_{n\to\infty}r_n$ exists.

Lemma 2.2. [9] Suppose that X is a uniformly convex Banach space and $0<p\leq t_n\leq q<1$ for all $n\in N$. Suppose further that $\{x_n\}$ and $\{y_n\}$ are sequences of X such that $\limsup_{n\to\infty}\left\|x_n\right\|\leq r,$ $\limsup_{n\to\infty}\left\|y_n\right\|\leq r,$ and $\limsup_{n\to\infty}\left\|t_n x_n+(1-t_n)y_n\right\|=r$ s hold for some $r\geq 0$. Then $\lim_{n\to\infty}\left\|x_n-y_n\right\|=0$.

We recall that a Banach space X is said to satisfy Opial's condition [12] if for any sequence $\{x_n\}$ in $X, x_n \to x$ implies that $\limsup_{n\to\infty}\|x_n - x\| < \limsup_{n\to\infty}\|x_n - y\|$ for all $y \in X$ with $y \neq x$. Moreover, we also know that a mapping $T : D \to X$ is called demiclosed with respect to $y \in X$ if for each sequence $\{x_n\}$ in D and each $x \in X, x_n$ and $Tx_n \to y$ imply that $x \in D$ and $Tx = y$.

Lemma 2.3. [11] Let X be a uniformly convex Banach space satisfying Opial's condition and let D be a nonempty of D into itself. Then I-t is demiclosed with respect to zero.

We shall now prove the following lemmas on the lines similar to [3]. It will be used to prove the main results.

Lemma 2.4. Let D be a nonempty closed convex bounded subset of normed space X and let $T, S : D \to D$ be two uniform $(L-\alpha)$-Lipschitz mappings. Define a sequence $\{x_n\}$ as in (1.3). Then

$$\|x_{n+1} - Tx_{n+1}\| \le d_{n+1} + L\left\{2d_n + Ld'^{\alpha}_n + L\left(d_n + Ld'^{\alpha}_n\right)^{\alpha}\right\}^{\alpha}$$

and

$$\|x_{n+1} - Sx_{n+1}\| \le d'_{n+1} + L\left\{d_n + Ld'^{\alpha}_n + d'_{n+1} + L\left(d_n + Ld'^{\alpha}_n\right)^{\alpha}\right\}^{\alpha}$$

where $d_n = \|x_n - T_n x_n\| and\, d'_n = \|x_n - S^n x_n\|$

Proof. We consider

$$\begin{aligned}
\|x_n - x_{n+1}\| &= \|x_n - \{(1-\alpha_n)x_n + \alpha_n T^n y_n\}\| \\
&\le \|x_n - T^n y_n\| \\
&\le \|x_n - T^n x_n\| + \|T^n x_n - T^n y_n\| \\
&\le d_n + L\|x_n - y_n\|^{\alpha} \\
&\le d_n + L\|x_n - \{(1-\beta_n)x_n + \beta_n S^n x_n\}\|^{\alpha} \\
&\le d_n + L\|x_n - S^n x_n\|^{\alpha}
\end{aligned}$$

$$(2.1)\qquad \le d_n + Ld'^{\alpha}_n$$

and

$$\begin{aligned}
\|xn+1 - Txn+1\| &\le \|x_{n+1} - T^{n+1}x_{n+1}\| + \|Tx_{n+1} - T^{n+1}x_{n+1}\| \\
&\le d_n + 1L\|x_n + 1 - T^n x_{n+1}\|^{\alpha} \\
&\le d_{n+1} + 1L\|(x_{n+1} - x_n) + x_n + (x_n - T^n x_n) + (T^n x_n - T^n x_{n+1})\|^{\alpha} \\
&\le d_{n+1}L\{\|x_{n+1} - x_n\| + \|x_n - T^n x_n\| + \|T^n x_n - T^n x_{n+1}\|\}^{\alpha} \\
&\le d_{n+1}L\left\{d_n + Ld'^{\alpha}_n + d_n + L\|x_n - x_{n+1}\|^{\alpha}\right\}^{\alpha} \\
&\le d_{n+1}L\left\{2d_n + Ld'^{\alpha}_n + L\left(d_n + Ld'^{\alpha}_n\right)^{\alpha}\right\}^{\alpha}
\end{aligned}$$

Similarly, we can prove that

$$\|x_{n+1} - Sx_{n+1}\| \le d'_{n+1} + L\left\{d_n + Ld'^{\alpha}_n + d'_n + L\left(d_n + Ld'^{\alpha}_n\right)^{\alpha}\right\}^{\alpha}$$

Lemma 2.5. Let X be a uniformly Banach space and let D be a nonempty closed convex bounded subset of X. Let $T,S:D\rightarrow D$ be a continuous mappings satisfying condition (1.2). Given a sequence $\{x_n\}$ defined by (1.3). Then

$$\|x_{n+1}-Tx_{n+1}\|\leq\frac{1-c_n}{1-3c_n}\left[d'_{n+1}\frac{a_n+c_n}{1-c_n}\left(\frac{1+a_n+2c_n}{1-c_n}.\frac{1+a_n}{1-c_n}d'_n+\frac{1+c_n}{1-c_n}d_n\right)\right]$$

and

$$\|x_{n+1}-Sx_{n+1}\|\leq\frac{1-c_n}{1-3c_n}\left[d_{n+1}+\frac{a_n+c_n}{1-c_n}\left(\frac{1+a_n+2c_n}{1-c_n}\frac{1+a_n}{1-c_n}d'_n+\frac{1-c_n}{1-c_n}d'_n\right)\right]$$

where $d_n=\|x_n-T^nx_n\| and\, d'_n=\|x_n-S^nx_n\|$.

Proof. We have

$$\|x_n-x_{n+1}\|=\|x_n-\{(1-a_n)x_n+a_nT^ny_n\}\|$$
$$\leq\|x_n-T^ny_n\|$$

(2.2) $$\leq\|x_n-S^nx_n\|+\|S^nx_n-T^ny_n\|$$

From condition (1.2). we have

$$\|S^nx_n-T^ny_n\|\leq a_n\|x_n-y_n\|+c_n(\|x_n-T^ny_n\|+\|y_n-S^nx_n\|)$$

$$\leq a_n\|x_n-y_n\|+c_n(\|x_n-S^nx_n\|+\|S^nx_n-T^ny_n\|)$$
$$+\|y_n-x_n\|+\|x_n-S^nx_n\|$$

Then

(2.3) $$\|S^nx_n-T^ny_n\|\leq\frac{a_n+c_n}{1-c_n}\|x_n-y_n\|+\frac{2c_n}{1-c_n}\|x_n-S^nx_n\|.$$

From (2.2) and (2.3), we obtain

$$\text{s}\|x_{n+1}-x_n\|\ \leq\ \|x_n-S^nx_n\|+\frac{a_n+c_n}{1-c_n}\|x_n-y_n\|+\frac{2c_n}{1-c_n}\|x_n-S^nx_n\|$$

(2.4) $$\leq\ \frac{1+a_n+2c_n}{1-c_n}d'_n$$

$$\|xn+1-Txn+1\|\ \leq\ \|xn+1-S^{n+1}x_{n+1}\|+\|Tx_{n+1}-S^{n+1}x_{n+1}\|$$
$$\leq\ d'_{n+1}+\frac{a_n+c_n}{1-c_n}\|x_{n+1}-S^nx_{n+1}\|$$
$$+\frac{2c_n}{1-c_n}\|x_{n+1}-Tx_{n+1}\|$$

$$\left(\frac{1-3c_n}{1-c_n}\right)\|x_{n+1}-Tx_{n+1}\|\ \leq\ d'_{n+1}+\frac{a_n+c_n}{1-c_n}(\|x_{n+1}-x_n\|+\|x_n-T^nx_n\|)$$
$$+\|T^nx_n-S^nx_{n+1}\|$$
$$\leq\ d'_{n+1}+\frac{a_n+c_n}{1-c_n}\left(\|x_{n+1}-x_n\|+d_n+\frac{a_n+c_n}{1-c_n}\|x_n-x_{n+1}\|\right.$$

$$+\frac{2c_n}{1-c_n}\|x_n - T^n x_n\|\Big)$$

(2.5) $$\leq d'_{n+1} + \frac{a_n + c_n}{1-c_n}\left[\frac{1+a_n}{1-c_n}\|x_n - x_{n+1}\| + \frac{1+c_n}{1-c_n}d_n\right]$$

Substituting (2.40) into (2.5), we get

$$\|x_{n+1} - Tx_{n+1}\| \leq \frac{1-c_n}{1-3c_n}\left[d'_{n+1} + \frac{a_n + c_n}{1-c_n}\left(\frac{1+a_n 1+a_n+2c_n}{1-c_n-1-c_n}d'_n + \frac{1+c_n}{1-c_n}d'_n\right)\right]$$

Similarly, we can prove that

$$\|x_{n+1} - S_{n+1}\| \leq \frac{1-c_n}{1-3c_n}\left[d'_{n+1} + \frac{a_n + c_n}{1-c_n}\left(\frac{1+a_n}{1-c_n}\frac{1+a_n+2c_n}{1-c_n}d'_n + \frac{1+c_n}{1-c_n}d'_n\right)\right]$$

Lemma 2.6. Let X be a Banach space X and D a nonempty subset of X. Let $S, T : D \to D$ be two mappings such that

$$\|T^n x_n - S^n y\| \leq a_n\|x-y\| + c_n\left(\|x - T^n y\| + \|y - S^n x\|\right)$$

for all $x, y \in D$ and $n \in N$, where $a_n, c_n \geq 0$ and satisfying the following condition:

(i) $c_n < 1 \; for all n \in N$

(ii) $\frac{a_n + 2c_n}{1-c_n} \leq 1 \; for all n \in N$

then

$$\|T^n x - p\| \leq \frac{a_n + c_n}{1-c_n}\|x - p\|.$$

and

$$\|S^n x - p\| \leq \frac{a_n + c_n}{1-c_n}\|x - p\| \text{ for all } x \in D \text{ and } n \in N.$$

Proof. (a) It follows from

Let $p \in F(T) \cap F(S)$ .Then

$$\|T^n x - p\| \leq a_n\|x-p\| + c_n\left(\|x-p\| + \|S^n x - p\|\right)$$
$$\leq (a_n + c_n)\|x-p\| + c_n\|S^n x - p\|$$

and

$$\|S^n x - p\| \leq (a_n + c_n)\|x-p\| + c_n\|S^n x - p\|.$$

Now

$$\|T^n x - p\| \leq (a_n - c_n)\|x-p\| + c_n\left[(a_n + c_n)\|x-p\| + c_n\|T^n x - p\|\right]$$
$$\leq (1+c_n)(a_n + c_n)\|x-p\| + c_n^2\|T^n x_n - p\|$$

which implies that

$$\|T^n x - p\| \leq \frac{a_n + c_n}{1-c_n}\|x-p\|.$$

Similarly,

$$\|S^n x_n - p\| \leq \frac{a_n + c_n}{1-c_n}\|x-p\| \text{ for all } x \in D \text{ and } n \in N.$$

3.MAIN RESULTS

Theorem 3.1. Let X be a uniformly convex Banach space and D be a nonempty closed convex bounded subset of Banach space X . Let $T, S : D \to D$ be an asymptotically quasi-nonexpansive mappings with sequence $\{k_n\}$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ and $F(T) \cap F(S) \neq \phi$. Define a sequence $\{x_n\}$ in D as (1.3) Then the following hold:

(a) $\lim_{n\to\infty}\|x_n - p\| = \lim_{n\to\infty}\|y_n - p\|$ exists.

(b) $\lim_{n\to\infty}\|x_n - Tx_n\| = 0 = \|x_n - Sx_n\|$ $if\ S\ and\ T\ is\ a\ uniform(L-a) - Lipschitizion.$

Proof. (a) Let $p \in F(T) \cap F(S)$. Then

$$\|x_{n+1} - p\| = \|(1 - a_n)x_n + a_n T^n y_n - p\|$$

$$\leq \|x_n - p\| + \|T^n y_n - p\|$$

(3.1) $$\leq \|x_n - p\| + k_n\|y_n - p\|$$

$$\|y_n - p\| = \|(1 - \beta_n)x_n + \beta_n S^n x_n - p\|$$

$$\leq (1 - \beta_n)\|x_n - p\| + \beta_n\|S^n x_n - p\|$$

(3.2) $$\leq k_n\|x_n - p\|$$

From (3.1) and (3.2), we get

$$\|x_{n+1} - p\| \leq k_n^2\|x_n - p\|$$

(b) Suppose $\lim_{n\leftarrow\infty}\|x_n - p\| = d$, for some d>0.

Since

$$\|T^n y_n - p\| \leq k_n\|y_n - p\|.$$

It follows that

$$\limsup_{n\to\infty}\|T^n y_n - p\| \leq \limsup_{n\to\infty}(k_n\|y_n - p\|)$$

We observe that

$$\lim_{n\to\infty}\|x_n - p\| \leq d\ and \lim_{n\to\infty}(T^n y_n - p) \leq d.$$

Then

$$\lim_{n\to\infty}\|x_{n+1} - p\| = \lim_{n\to\infty}\|a_n(T^n x_n - p) + (1 - a_n)(x_n - p)\| - d$$

From Lemma 2.2, we obtain

(3.3) $$\lim_{n\to\infty}\|x_n - T^n y_n\| = 0.$$

Further

$$\|x_n - p\| \leq \|x_n - T^n y_n\| + \|T^n y_n - p\|$$

$$\leq \|x_n - T^n y_n\| + k_n\|y_n - p\|$$

Gives that

$$d \leq \liminf_{n\to\infty}\|y_n - p\| \leq \limsup_{n\to\infty}\|y_n - p\| \leq d.$$

Hence

$$\lim_{n\to\infty}\|y_n - p\| - d,$$

Implies that

$$\lim_{n\to\infty}\left\|\beta_n\left(S^n x_n - p\right) + \left(1-\beta_n\right)\left(x_n - p\right) = d\right\|$$

Using Lemma 2.2, we get

(3.4) $$\lim_{n\to\infty}\left\|x_n - S^n x_n\right\| = 0.$$

Again

$$\begin{aligned}\left\|T^n x_n - x_n\right\| &\leq \left\|T^n x_n - T^n y_n\right\| + \left\|T^n y_n - x_n\right\| \\ &\leq k_n\left\|x_n - y_n\right\| + \left\|T^n y_n - x_n\right\| \\ &\leq k_n\left\|x_n - \left\{\left(1-\beta_n\right)x_n + \beta_n S^n x_n\right\}\right\| + \left\|T^n y_n - x_n\right\| \\ &\leq k_n\left\|x_n - S^n x_n\right\| + \left\|T^n y_n - x_n\right\|\end{aligned}$$

Implies together with (3.3) and (3.4) that

$$\lim_{n\to\infty}\left\|x_n - T^n x_n\right\| = 0 \ \lim_{n\to\infty}\left\|x_n - S^n x_n\right\|$$

Applying Lemma 2.4 shows that

$$\lim_{n\to\infty}\left\|x_n - Tx_n\right\| = 0 \ \lim_{n\to\infty}\left\|x_n - Sx_n\right\|$$

Theorem 3.2. Let X a uniformly convex Banach space satisfying Opial's condition and let D, T, S and $\{x_n\}$ be as taken in Theorem 3.1. If $F(T)\bigcap F(S) \neq \theta$ then $\{x_n\}$ converges weakly to a common fixed point of T and. S.

Proof. We prove that $\{x_n\}$ has a unique weak subsequential limit in $F(T)\bigcap F(S)$. To prove this, let υ and υ be weak limits of the subsequences $\{x_{nj}\}$ and $\{x_{nj}\}$ of $\{x_n\}$respectively. By Theorem 3.1, $\lim_{n\to\infty}\left\|x_n - Tx_n\right\| = 0 \ \lim_{n\to\infty}\left\|x_n - Sx_n\right\|$ and I-T,I-S are demiclosed with respect to zero by Lemma 2.3, we obtain that $Tu = u$ and $Su = u$. Similarly, we can prove that $u \in F(T)\bigcap F(S)$. If $u \neq \upsilon$, then by Opial's condition.

$$\begin{aligned}\lim_{n\to\infty}\left\|x_n - u\right\| = \lim_{n\to\infty}\left\|x_n - u\right\| &< \lim_{n\to\infty}\left\|x_{ni} - u\right\| \\ &= \lim_{n\to\infty}\left\|x_n - \upsilon\right\| = \lim_{n\to\infty}\left\|x_{nj} - u\right\| \\ &< \lim_{n\to\infty}\left\|x_{nj} - u\right\| = \lim_{n\to\infty}\left\|x_n - u\right\|\end{aligned}$$

This is contradiction and hence the proof is complete.

Theorem 3.3. Let X be uniformly convex Banach space and let D be a nonempty closed convex bounded subset of X which satisfying Opial's condition. Let $T, S : D \to D$ be a continuous mappings satisfying condition (1.2). Given a sequence as in (1.3), then $\{x_n\}$ converges weakly to a common fixed point of T and S.

Proof. Since $p \in F(T)\bigcap F(S)$. Then

$$\begin{aligned}\left\|x_{n+1} - p\right\| &= \left\|\left(1-a_n\right)x_n + a_n T^n y_n - p\right\| \\ &\leq \left\|x_n - p\right\| + \left\|T^n y_n - p\right\|\end{aligned}$$

Using Lemma 2.6, we obtain

$$\begin{aligned}\left\|x_{n+1} - p\right\| &\leq \left\|x_n - p\right\| + \frac{a_n + c_n}{1-c_n}\left\|p - y_n\right\| \\ &\leq \left\|x_n - p\right\| + \frac{a_n + c_n}{1-c_n}\left(\left\|p - x_n\right\| + \left\|p - S^n x_n\right\|\right)\end{aligned}$$

New, again using Lemma (2.6)

$$\|x_{n+1}-p\| \leq \|x_n-p\| + \frac{a_n+c_n}{1-c_n}\|x_n-p\| + \left(\frac{a_n+c_n}{1-c_n}\right)^2\|x_n-p\|$$

Let $\frac{a_n+c_n}{1-c_n} = L_n$. Then

$$\|x_{n+1}-p\| \leq (1+L_n+L_n^2)\|x_n-p\|$$

From Lemma 2.1, we get $\lim_{n\to\infty}\|x_n-p\|$ exists.

Let $\lim_{n\to\infty}\|x_n-p\| = d$ for some d>0.

Since

$$\|y_n-p\| = \|(1-\beta_n)x_n+\beta_n S^n x_n - p\|$$
$$\leq \|x_n-p\| + L_n\|x_n-p\|.$$

Now

$$\limsup_{n\to\infty}\|y_n-p\| \leq \limsup_{n\to\infty}\|x_n-p\| \leq d.$$

and

$$\|T^n y_n - p\| = L_n\|y_n-p\|$$

Then

$$\limsup_{n\to\infty}\|T^n y_n - p\| \leq \limsup_{n\to\infty}\|x_n-p\| \leq d.$$

Now

$$\limsup_{n\to\infty}\|y_n-p\| \leq \limsup_{n\to\infty}\|x_n-p\| \leq d.$$

Now consider, we have

$$\lim_{n\to\infty}\|x_{n+1}-p\| = \lim_{n\to\infty}\|a_n(T^n y_n - p) + (1-a_n)(x_n-p)\|.$$

From Lemma 2.2, we obtain

$$\lim_{n\to\infty}\|x_n - T^n x_n\| = 0.$$

Next

$$\|x_n-p\| \leq \|x_n - T^n y_n\| + \|T^n y_n - p\|$$
$$\leq \|x_n - T^n y_n\| + L_n\|y_n-p\|.$$

Note that

$$\|xn-p\| \leq \liminf_{n\to\infty}\|yn-p\| \leq \limsup_{n\to\infty}\|y_n-p\| \leq d.$$

Hence

$$\lim_{n\to\infty}\|y_n-p\| = d.$$

That is

$$\lim\|\beta_n(S^n x_n - p) + (1-\beta_n)(x_n-p)\| = d.$$

Since

$$\limsup_{n\to\infty}\|S^n x_n - p\| \leq d \; and \limsup_{n\to\infty}\|x_n-p\| \leq d.$$

From Lemma 2.2, we obtain

(3.5) $$\lim_{n\to\infty}\|x_n - S^n x_n\| = 0.$$

Now, again

$$\limsup_{n\to\infty}\|S^n x_n - p\| \le d \, and \limsup_{n\to\infty}\|x_n - p\| \le d.$$

From Lemma 2.2, we get

(3.6) $$\lim_{n\to\infty}\|x_n - S^n x_n\| = 0.$$

We obtain that

$$\|x_n - T^n x_n\| \le \|x_n - S^n y_n\| + \|S^n y_n - T^n x_n\|$$
$$\le \|x_n - S^n y_n\| + L_n\|x_n - y_n\| + \frac{2c_n}{1-c_n}\|T^n x_n\|$$
$$\le \frac{1-c_n}{1-3c_n}\left\{L_n\|x_n - S^n x_n\| + \|x_n - S^n y_n\|\right\}$$

Implies together with (3.5) and (3.6) that

$$\lim_{n\to\infty}\|x_n - S^n x_n\| = 0 = \lim_{n\to\infty}\|x_n - T^n x_n\|.$$

Lemma 2.5 reveals that

$$\lim_{n\to\infty}\|x_n - Tx_n\| = 0 = \lim_{n\to\infty}\|x_n - Sx_n\|$$

The rest of the proof follows the lines similar to Theorem 3.2 and is therefore omitted. This completes the proof of the theorem.

Theorem 3.4. Let D be a nonempty compact convex subset of a uniformly convex Banach space X and T,S and $\{x_n\}$ as in Theorem 3.1. If $F(T)\cap F(S)\ne\theta$, then $\{x_n\}$ converges strongly to a common fixed point of T and S.

Theorem 3.5. Let D be a nonempty compact convex subset of a uniformly convex Banach space X and let $T,S : D \to D$ be a continuous mappings satisfying condition (1.1). Given a sequence $\{x_n\}$ as in (1.3), the $\{x_n\}$ converges strongly to a common fixed point of T and S.

Remark 3.6 Theorem 3.4 and Theorem 3.5 generlaize the results of Khan and Takahashi [3, Theorem 2].

CONCLUSIONS

The study of the approximation of common fixed points for more general classes of mappings through weak and strong convergence results of an iterative scheme in a uniformly convex Banach space shows that $T,S : D \to D$ be an asymptotically quasi-nonexpansive mappings with sequence $\{k_n\}$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ and $F(T)\cap F(S)\ne\phi$.

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