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Coupled Pendulums: A Classical Depiction of Laser Oscillation, Phase Diffusion and Laser Line-width

[1]Rajib Bordoloi, [2]Ranjana Bora Bordoloi & [3]Gauranga Dhar Baruah
[1] Women's College, Tinsukia – 786125, Assam, India.
[2] Physics Dept., Namrup College, Namrup – 786623, Assam, India.
[3] Centre for Laser and Optical Science, New Uchamati,
Doomdooma – 786157, Assam, India.

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ABSTRACT

In this work we have presented an analogy between the coupled vibrations of two classical oscillators and the oscillations that take place in a laser cavity. Our aim is to understand classically the causes that lead to the phase diffusion in a system of coupled classical oscillators and to explore possibilities of any relationship between phase fluctuation and the frequency difference. The equations of motion for the classical oscillators have been derived and solved, for different values of coupling coefficients, to obtain the expressions for the mode frequencies[1]. The solutions, while plotted graphically have led us to the conclusion that in classical oscillators the mode frequencies of the oscillators are far apart if their oscillation is heavily coupling dependent and consequently the phase relationship of the oscillators fluctuate vigorously and frequently, which is the converse of what happens in a laser cavity consisting atomic oscillators.

Key words: Coupled pendulum, Mode frequency, Phase diffusion, Laser line width

1. Introduction:

The line width of a laser is a measure of its phase noise. The spectrogram of a laser is produced by passing its light through a prism. The spectrogram of the output of a pure noise-free laser will consist of a single infinitely thin line. However, if the laser exhibits phase noise, the line will have non-zero width. The greater the phase noise, the wider the line. The same will be true with oscillators. The spectrum of the output of a noise-free oscillator has energy at each of the harmonics of the output signal, but the bandwidth of each harmonic will be zero. If the oscillator exhibits phase noise, the harmonics will not have zero bandwidth. The more phase noise the oscillator exhibits, the wider the bandwidth of each harmonic. With a view to establishing an analogy between the phase noise in a laser cavity, which is of quantum mechanical origin, and the one that occurs in a coupled oscillator system, we will consider the energy exchange between classical oscillators. To begin with we are going to consider two simple harmonic pendulums coupled by spring of spring constant *k*. The result obtained for two oscillators can be later extended to any number of oscillators.

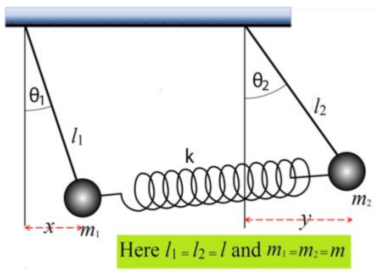

Fig.1: Two identical pendulums coupled by a spring of constant k.

2. Coupled vibration of Classical oscillators:

Let us consider two pendulums with bobs each having a mass *m* and length *l* are coupled by springs of constant *k*. Let the two bobs suffer displacements *y* and *x* respectively ($y > x$). Assuming that the spring obeys Hooke's law, the return force on the masses is $F_1 = k(y-x)$. For the sake of linearity, we take *y* and *x* to be small. Hence the restoring forces due to gravity acting on the bobs m_1 and m_2 are $F_{2x} = \frac{mg}{l}\tan\theta_1 = \frac{mg}{l}x$ and $F_{2y} = \frac{mg}{l}\tan\theta_2 = \frac{mg}{l}y$ respectively. Hence the equations of motion are [1]:

$$m\ddot{x} = -\frac{mgx}{l} + k(y-x) \tag{1}$$

$$m\ddot{y} = -\frac{mgy}{l} + k(y-x) \tag{2}$$

To solve the above two equations let us rewrite them as

$$-\ddot{x}-\left(-\frac{g}{l}-\frac{k}{m}\right)x+\frac{k}{m}y=0 \tag{3}$$

$$\frac{k}{m}x-\ddot{y}+\left(-\frac{g}{l}-\frac{k}{m}\right)y=0 \tag{4}$$

Equations (3) and (4) can be written in matrix form as

$$\begin{pmatrix} -\frac{d^2}{dt^2}-\left(\frac{g}{l}+\frac{k}{m}\right) & \frac{k}{m} \\ \frac{k}{m} & -\frac{d^2}{dt^2}-\left(\frac{g}{l}+\frac{k}{m}\right) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

or, we can write

$$AX=0 \tag{5}$$

with

$$A=\begin{pmatrix} -\frac{d^2}{dt^2}-\left(\frac{g}{l}+\frac{k}{m}\right) & \frac{k}{m} \\ \frac{k}{m} & -\frac{d^2}{dt^2}-\left(\frac{g}{l}+\frac{k}{m}\right) \end{pmatrix}, \text{and } X=\begin{pmatrix} x \\ y \end{pmatrix}$$

Equ. (5) implies that either

$$X=0 \tag{6}$$

$$\text{or, } |A|=\left|\begin{pmatrix} -\frac{d^2}{dt^2}-\left(\frac{g}{l}+\frac{k}{m}\right) & \frac{k}{m} \\ \frac{k}{m} & -\frac{d^2}{dt^2}-\left(\frac{g}{l}+\frac{k}{m}\right) \end{pmatrix}\right|=0 \tag{7}$$

, or both must be solutions of (5). In equations (3) & (4), by general intuition we can consider $x=X_0e^{i\omega t}$, similarly $y=Y_0e^{i\omega t}$. This leads us to $\ddot{x}=-\omega^2 x$, similarly $\ddot{y}=-\omega^2 y$, which in turn converts equation (7) to

$$|A|=\left|\begin{pmatrix} \omega^2-\left(\frac{g}{l}+\frac{k}{m}\right) & \frac{k}{m} \\ \frac{k}{m} & \omega^2-\left(\frac{g}{l}+\frac{k}{m}\right) \end{pmatrix}\right|=0 \tag{8}$$

This equation gives us the Eigen value equation as

$$\omega^2-\left(\frac{g}{l}+\frac{k}{m}\right)=\pm\frac{k}{m} \tag{9}$$

This shows that there are two Normal Mode frequencies, corresponding to the two independent solutions of the coupled differential equations (3) and (4), given by

$$\omega_1 = \sqrt{g/l} \tag{10-a}$$
$$\omega_2 = \sqrt{g/l + 2k/m} \tag{10-b}$$

Here the ± sign is ignored as it gives rise to the same sinusoidal solutions. Now to find the normal modes, we solve equation (5-a) for X by substituting each normal mode frequencies in turn. For $\omega = \omega_1$ we have

$$\begin{pmatrix} +\frac{k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & +\frac{k}{m} \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{11}$$

Which gives $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = A_1 e^{i\phi_1} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ (12)

Similarly for $\omega = \omega_2$ we have

$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = A_2 e^{i\phi_2} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{13}$$

Finally, since $X = \mathrm{Re}(x.e^{i\omega t})$ we have the two "normal mode" solutions

$$X_{1,2} = \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix} A_{1,2} \cos(\omega_{1,2} t + \phi_{1,2}) \tag{14}$$

Using the principle of linear superposition we can write the general solution as the linear combination of the two normal mode solutions

$$X = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} A_1 \cos(\omega_1 t + \phi_1) + \begin{pmatrix} 1 \\ -1 \end{pmatrix} A_2 \cos(\omega_2 t + \phi_2) \tag{15}$$

Using the formula for $\cos\theta_1 + \cos\theta_2$ we get,

$$x(t) = A\cos\left(\frac{(\omega_1 + \omega_2)t}{2}\right)\cos\left(\frac{(\omega_1 - \omega_2)t}{2}\right) \tag{16}$$

and $$y(t) = A\sin\left(\frac{(\omega_1 + \omega_2)t}{2}\right)\sin\left(\frac{(\omega_1 - \omega_2)t}{2}\right) \tag{17}$$

3. **Results and Discussion:**

The Eqns. (10), (16) and (17) show that the mode frequencies[1] ω_1 & ω_2 and hence the solutions show a dependence on the masses of the oscillators and their coupling coefficients. Let us make a comparative analysis of the solutions for three different sets of coupling coefficients (keeping the lengths and the masses of the pendulums unchanged), both analytically and graphically.

Case–1: l = 0.5, k = 1, m =1

This gives us the mode frequencies as $\omega_1 = \sqrt{g/0.5} = 4.43\,\text{rad/sec}$ and $\omega_2 = 4.65\,\text{rad/sec}$. Also the mean oscillating frequency and the modulating frequency are $\bar{\omega} = 4.54\,\text{rad/sec}$ and $\omega_M = (4.65 - 4.43)/2 = 0.11\,\text{rad/sec}$ respectively

Case – 2: l = 0.5, k=6, m =1

This gives us the mode frequencies as $\omega_1 = \sqrt{g/0.5} = 4.43\,\text{rad/sec}$, $\omega_2 = 5.62\,\text{rad/sec}$, the mean oscillating frequency $\bar{\omega} = 5.02\,\text{rad/sec}$ and the modulating frequency and $\omega_M = 0.59\,\text{rad/sec}$ respectively.

Case – 3: l=0.5, k=16, m =1

This gives us the mode frequencies as $\omega_1 = \sqrt{g/0.5} = 4.43\,\text{rad/sec}$, $\omega_2 = 7.19\,\text{rad/sec}$, the mean oscillating frequency $\bar{\omega} = 5.81\,\text{rad/sec}$ and the modulating frequency and $\omega_M = 1.38\,\text{rad/sec}$ respectively.

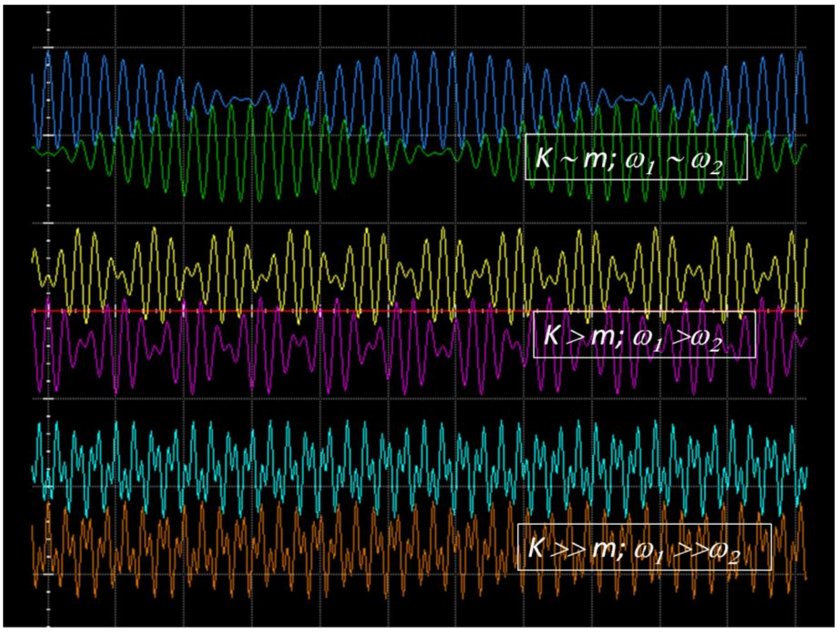

Fig. 2: Energy exchange between two coupled pendulums. It can be seen that the "Phase Diffusion" is an inverse function of the "Frequency Spread" between the Normal Modes.

4. Conclusion:

We have derived the equations of motion for the two oscillating bodies (pendulums) and solved them using matrix method. The two solutions, when plotted graphically for different values of coupling constants, provide us with a clear picture on the nature of energy exchange between the coupled oscillators. A close look at the plots leads us to a conclusion that if the system is controlled by very strong coupling force (k>>m), the two *mode frequencies* are far apart, consequently the energy transfer is not complete and the phase fluctuation is very high and both phase and the amplitude fluctuations are also much frequent. If the coupling factors are much less or of the order of the inertial coefficients (mass), the two mode frequencies are very close to each other, consequently the energy transfer is complete and, the phase fluctuation is negligible and the amplitude fluctuation takes place very slowly. If we generalize it to a system of N oscillators, then there would be N coupled modes, N coupled equations with all sorts of coefficients on the right hands side. By forming linear combinations we can find a set of normal modes Y_i with normal frequencies ω_i. Another important point we have observed is that in classical coupled oscillations the normal mode frequencies are less influenced by variation of mass of the oscillators in comparison to that of the coupling constants. i.e. the oscillations are coupling controlled.

This behaviour of classical coupled oscillators bears a striking similarity with what happens in a laser cavity [4-6], where the atomic oscillators interact among themselves to generate line width of a laser. We know that in a laser cavity the phase diffusion occurs due to various classical and quantum mechanical factors (e.g. spontaneous emission), which consequently causes a spread in the frequency of the laser line, i.e. the line width. On the other hand in a system of classical oscillators exactly the converse effect takes place, i.e. the difference in mode frequencies (frequency spread) leads to phase fluctuation (phase diffusion). Another point to keep note is that the spread in the mode frequency in classical oscillators is a direct function of the coupling constants.

Applying the above findings directly to the physics of the laser cavity, we can conclude that to reduce phase noise in a laser cavity we need to reduce the coupling effects among the atomic oscillators. One way to do this is to reduce the number density inside the cavity/ or may be to increase the equilibrium temperature in the cavity.

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