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L - Fuzzy BP – Algebras

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ABSTRACT

In this paper, we define the notion of L-Fuzzy BP-Algebras. We discuss the properties of L-Fuzzy BP-subalgebras and prove results on the notion of Intersection of L-fuzzy BP-subalgebras and Cartesian product of L- fuzzy BP-subalgebras.

Key words: BP-algebra, Fuzzy BP-algebra, L-Fuzzy BP-algebra

1. Introduction: In 1966 Y.Imai and K.Iseki introduced two classes of abstract algebra, BCK algebras and BCI algebras [3,4]. In 2012 Sun Shin Ahn and Jeong Soon Han introduced the notion of BP-Algebras [6]. In 1971 A.Rosenfeld initiated the study of fuzzy algebraic structures [5] In 1965 L.A.Zadeh introduced the notion of fuzzy sets [7]. L Goguen extended the notion of fuzzy sets into L-fuzzy sets where L is a complete lattice [2]. In our earlier paper we have introduced the notion of fuzzy structures in BP-algebras [1]. In this paper, we introduce the notion of L-Fuzzy BP-Algebras.

2. Preliminaries

In this section we recall some basic definitions that are needed for our work.

Definition 2.1 A BP- algebra $(X, *, 0)$ is a non-empty set X with a constant 0 and a binary operation $*$ satisfying the following conditions:

1. $x * x = 0$
2. $x *(x * y) = y$
3. $(x *z) *(y *z) = x *y)$, for any $x, y, z \in X$

Definition 2.2 Let S be a non-empty set. A mapping $\mu : S \to [0, 1]$ is called a fuzzy subset of S.

Definition 2.3 A lattice is a partially ordered set in which any two elements have a least upper bound and a greatest lower bound.

Definition 2.4 A lattice L is called a complete lattice if every subset $A = \{a_\alpha\}$ has a sup denoted by $\vee a_\alpha$ and inf denoted by $\wedge a_\alpha$ where $0 \equiv \wedge a_\alpha$ is the least element of L and $1 \equiv \wedge a_\alpha$ is the greatest element of L: $0 \leq a$ and $1 \geq a$ for every $a \in L$.

Definition 2.5 Let X be a non-empty set and $L:(L, \leq)$ be a complete lattice with least element 0 and greatest element 1. A L-fuzzy subset μ of X is a function $\mu: X \to L$.

3. L-Fuzzy BP-subalgebra

In this section we introduce the notion of L-Fuzzy BP-subalgebra. Throughout this section L denote complete Lattice.

Definition 3.1 :A L- fuzzy subset μ of a BP-algebra $(X,*,0)$ is called a L-fuzzy BP subalgebra of X if, for all $x,y \in X$ the following condition is satisfied

$$\mu(x* y) \geq \mu(x) \wedge \mu(y)$$

Example 3.2 Let X = {0,a, b, c } be a set with the following table

∗	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Then (X,∗, 0) is a BP – algebra

Define μ: X → L by $\mu(x) = \begin{cases} 1 \ \ if \ \ x = 0 \\ t_1 \ \ if \ \ x = b \\ t_2 \ if \ \ x = a \\ 0 \ \ if \ \ x = c \end{cases}$

t_1 , t_2 ϵ L and inf L≤ t_1 ≤ t_2 ≤ Sup L
Then μ is a L-fuzzy BP-subalgebra of X.

One can easily prove that:
Theorem 3.4 *Intersection of any two L-fuzzy BP-sub algebras of X is again a L fuzzy BP- sub algebra.*

Definition 3.5 *Let μ be any L-fuzzy subset of a BP – algebra (X,∗, 0) and let t ∈ L*
The set U(μ, t) = { x ∈ ***X*** *: μ(x) ≥ t }*
is called a level subset of μ of X.

Lemma 3.6 *Let (X,∗, 0) be a BP- sub algebra. Let* μ *be a L- fuzzy BP – subalgebra of X.*
Let α ∈ L. Then

1. *U(*μ*, α) is either ∅ or a BP- sub algebra of X*
2. μ(0) ≥ μ(x) *for all x ∈* ***X***

Proof:
For any α ∈ L, assume that U(μ, α) is non-empty .
Let x, y ∈ U(μ, α). Therefore μ*(x)* ≥ α, μ*(y)* ≥ α
To show that U(μ, α) is a BP – subalgebra, we need to show x∗y ∈ U(μ, α).
That is, we need to show

$$\mu(x * y) \geq \mu(x) \wedge \mu(y) \\ \geq \alpha \wedge \alpha \\ = \alpha$$

Also, μ(0) = μ(x∗ x) ≥μ(x) ∧ μ(x) = μ(x)
Since x∗ x = 0 ∀ x ∈ X
Thus μ(0) ≥ μ(x), ∀ x ∈ X

Lemma 3.7 *A L-fuzzy subset μ of a BP – subalgebra X is a L fuzzy BP- subalgebra if and only if for all t ∈ L, the level set of μ, U(μ, t) is either empty or a BP – subalgebra of X.*
Proof:
Assume that the level subset of μ in X,
U(μ, t) ≠ ∅
Then for any x, y ∈ U (μ, t),
μ(x) ≥ t, μ(y) ≥ t

Now, $\mu(x* y) \geq \mu(x) \wedge \mu(y) \geq t$
which implies $x* y \in U(\mu, t)$ and hence $U(\mu, t)$ is a BP – subalgebra of X.
Conversely assume that $U(\mu, t)$ is a BP- subalgebra of X
Take $t = \mu(x) \wedge \mu(y)$ for any $x, y \in X$
$x, y \in X$ *implies* $x* y \in U(\mu, t)$
Hence $\mu(x* y) \geq t = \mu(x) \wedge \mu(y)$, thus proving that μ is a L-fuzzy BP – subalgebra of X.

As in the case of Fuzzy BP algebra one can prove the following Lemma3.8 and Theorem 3.9.

Lemma 3.8 *Any BP – subalgebra of a BP- algebra (X,* , 0) can be realized as a level subalgebra of some L fuzzy BP-subalgebra of X*

Theorem 3.9 *Let A be a subset of X. Then the characteristic function* χ_A *is a L-fuzzy BP- subalgebra of X if and only if A is a BP- subalgebra of X*

Theorem 3.10 Let μ be a L-fuzzy BP- subalgebra of (X,∗, 0) with finite image. If $U(\mu, s) = U(\mu, t)$ for some $s, t \in Im(\mu)$, then $s = t$.
Proof:
Let μ be a L-fuzzy BP- subalgebra of X with finite image such that
$U(\mu, s) = U(\mu, t)$ for some $s, t \in Im(\mu)$.
Now, μ is a L-fuzzy algebra of X shows that $U(\mu, s)$ is a BP-subalgebra.
Therefore, if $x, y \in U(\mu, t) = U(\mu, s)$ then $\mu(x) \geq t$ and $\mu(y) \geq t$.
Also, $x, y \in U(\mu, t) = U(\mu, s)$ and $U(\mu, s)$ is a BP-subalgebra shows that $x* y \in U(\mu, s)$.
This shows that
$\mu(x* y) \geq \mu(x) \wedge \mu(y) \geq s$.
Thus we have, $\mu(x* y) \geq s$ as well as $\mu(x* y) \geq t$ whenever $x, y \in U(\mu, t) = U(\mu, s)$.
Similarly, we can prove that, $\mu(x* y) \geq s$ as well as $\mu(x* y) \geq t$ whenever $x, y \in U(\mu, s) = U(\mu, t)$.
This proves that $s = t$.

Lemma 3.11 *Let μ and λ be two L- fuzzy BP – sub algebras of X with identical family of level BP – sub algebras. If* $Im(\mu) = \{ t_1, t_2, \ldots t_n\}$ *and* $Im(\lambda) = \{s_1, s_2, \ldots, s_m\}$ *where F*
$t_1 \geq t_2 \geq \ldots \geq t_n$ *and* $s_1 \geq s_2 \geq, \ldots, \geq s_m$ *Then*

1. $m = n$
2. $U(\mu, t_i) = U(\lambda, s_i)$ *for* $i = 1, 2, \ldots n$
3. If $\mu(x) = s_i$, *then* $\lambda(x) = s_i, \forall x \in X$ *and* $i = 1, 2, \ldots n$

Proof:
Let μ and λ be two L-fuzzy BP – sub algebras of X with identical family of level BP – sub algebras $F(\mu) = F(\lambda)$.
Let $Im(\mu) = \{ t_1, t_2, \ldots t_n\}$ where $t_1 \geq t_2 \geq \ldots \geq t_n$ (1.1)
and $Im(\lambda) = \{s_1, s_2, \ldots, s_m\}$ where $s_1 \geq s_2 \geq, \ldots, \geq s_m$ (1.2)
(1.1) *implies* $U(\mu, t_1) \subseteq U(\mu, t_2) \subseteq \ldots \subseteq U(\mu, t_n) = X$ (1.3)
(1.2) implies $U(\lambda, s_1) \subseteq U(\lambda, s_2) \subseteq \ldots \subseteq U(\lambda, s_n) = X$ (1.4)
and $F(\mu) = \{U(\mu, t_i) : 1 \leq i \leq n \}$,
$F(\lambda) = \{U(\lambda, s_i) : 1 \leq j \leq m \}$
Assume $m \neq n$.
Then, $m \geq n$ or $n \geq m$.
Let $m \geq n$.
Then $U(\mu, t_i) = U(\lambda, s_i), i = 1, 2, \ldots n$.
This shows that both t_i *and* $s_i \in Im(\mu)$.
For $i > n$ we observe that $t_i \notin Im(\mu)$ and hence,

$U(\mu, t_i) \neq U(\lambda, s_i) i = n+1, n+2, \ldots m.$
Let n ≥ m. Then $U(\mu, t_i) = U(\lambda, s_i)$
$i = 1, 2, \ldots m.$ *This shows that both* t_i *and si* $\in Im(\lambda)$. For j> m we observe that $s_j \notin Im(\mu)$ and hence,
$U(\mu, t_i) \neq U(\lambda, s_i) i = m+1, m+2, \ldots n.$
(1.3) and (1.4) implies $t_i \neq, s_i,$
$\forall i = 1, 2, \ldots n$
Hence we can find some i such that $U(\mu, t_i) \neq U(\lambda, s_i)$.
This contradicts that F(μ) = F(λ).
Hence we conclude that m = n.

1. By part(1), we have proved that m = n. Since μ and λ have identical family of level sub algebras, we have
$$U(\mu, t_i) = U(\lambda, s_i), \quad i = 1, 2, \ldots n.$$

2. Follows from (1) and (2)
Let $\mu(x) = t_i$, *implies* $\lambda(x) = s_i$, *for* $i = 1, 2, \ldots, n$

Theorem 3.12*Let μ and λ be two L- fuzzy sub algebras of X with identical family of level sub algebras. Then Im(μ) = Im(λ) implies μ = λ*
Proof:
Let μ and λ be two L-fuzzy sub algebras of X with identical family of level sub algebras.
Let $Im(\mu) = Im(\lambda) = \{s_1, s_2, \ldots, s_n\}$
Where $s_1 \geq s_2 \geq, \ldots, \geq s_n$
By lemma 3.11 for any x∈ X, there exists s_i such that $\mu(x) = s_i = \lambda(x)$.
Thus $\mu(x) = \lambda(x)\ \forall x \in X$, *proving that* $\mu = \lambda$

Theorem 3.13 *Two level BP- sub algebras U(μ, s) and U(μ, t), (s<t) of a L fuzzy BP- subalgebra μ are equal if and only if there is no x∈ X such that* $s \leq \mu(x) < t$.
Proof:
Let U(μ, s) and U(μ, t) be two level BP-sub algebras of L-fuzzy BP-subalgebra μ of X
Suppose that U(μ, s) = U(μ, t) for some *s<t.*
Suppose there is one x∈ X such that $s \leq \mu(x) < t$.
Then, $\mu(x) \geq s$ and $\mu(x) < t$.
That is, x ∈U(μ, s) and x ∉U(μ, t).
This contradicts the fact that U(μ, s) = U(μ, t).

Conversely, assume that there is no x∈ X such that $s \leq \mu(x) < t$.
Suppose ,U(μ, s) ≠U(μ, t)
For, x∈ U(μ, t)$\Longrightarrow \mu(x) \geq t > s$
$\Longrightarrow \mu(x) > s \Longrightarrow x \in U(\mu, s)$
Since
U(μ, s) ≠U(μ, t), choose U(μ, s) ⊈U(μ, t).
Hence there is an x∈U(μ, s) and
x∉U(μ, t). $\Longrightarrow \mu(x) \geq s$ and $\mu(x) < t$.
Thus there exists an element x∈ X such that $s \leq \mu(x) < t$, this contradicts our hypothesis.
Hence U(μ, s) = U(μ, t).

Definition 3.14 *Let λ and μ be the L-fuzzy set in a set X. The Cartesian product*
$\lambda \times \mu : X \times X \to [0, 1]$ is defined by $(\lambda \times \mu)(x,y) = \{ \lambda(x) \wedge \mu(y)\}\ \forall x \in X$.

Theorem 3.15 *If μ_1 and μ_2 are L- fuzzy BP – sub algebras of X, then $\mu = \mu_1 x \mu_2$ is a L-fuzzy BP – subalgebra of X x X.*

Proof:

For any (x_1, x_2) and $(y_1, y_2) \in$ X x X, we have,

$$\begin{aligned}\mu((x_1, x_2)*(y_1, y_2)) &= \mu(x_1*y_1, x_2*y_2)\\ &= (\mu_1 x \mu_2)(x_1*y_1, x_2*y_2)\\ &= \mu_1(x_1*y_1) \wedge \mu_2(x_2*y_2)\\ &\geq (\mu_1(x_1) \wedge \mu_1(y_1)) \wedge (\mu_2(x_2) \wedge \mu_2(y_2))\\ &= (\mu_1(x_1) \wedge (\mu_2(x_2)) \wedge (\mu_1(y_1) \wedge \mu_2(y_2))\\ &= (\mu_1 x \mu_2)(x_1, x_2) \wedge (\mu_1 x \mu_2)(y_1, y_2)\\ &= \mu(x_1, x_2) \wedge \mu(y_1, y_2)\end{aligned}$$

Hence $\mu = \mu_1 x \mu_2$ is a L-fuzzy BP – subalgebra of X x X.

Definition 3.16 *Let $(X_1, *_1, 0_1)$ and $(X_2, *_2, 0_2)$ be BP- algebras. A mapping $f: X_1 \to X_2$ is called a homorphism if,*

$$f(x *_1 y) = f(x) *_2 f(y) \ \forall x, y \in X$$

Definition 3.17 *Let f be any function from the BP- algebra X_1 to the BP- algebra X_2. Let* μ be any fuzzy BP- subalgebra of X_1 satisfying supremum property and σ be any fuzzy BP – subalgebra of X_2. The image of μ under f, denoted by f(μ), is L- fuzzy subset of X_2 defined by

$$(f(\mu(y)) = \begin{cases} Sup_{x \in f^{-1}(Y)} \ \mu(x) & if \ f^{-1}(y) \neq \emptyset \\ 0 & otherwise \end{cases}$$

where $y \in X_2$. The pre image of σ under f, symbolized by $f^{-1}(\sigma)$, is a L-fuzzy subset of X_1 defined by

$$(f^{-1}(\sigma))(x) = \sigma(f(x)) \ \forall x, \in X_1.$$

Lemma 3.18 *Let $(X_1, *_1, 0_1)$ and $(X_2, *_2, 0_2)$ be two BP- algebras. Let $f: X_1 \to X_2$ be an epimorphism. If σ is L fuzzy BP- subalgebra of X_2, then* $f^{-1}(\sigma)$ is a L-fuzzy BP- subalgebra of X_1.

Alternatively, we have epimorphic pre image of a L- fuzzy BP- subalgebra is a L-fuzzy BP- sub algebra.

Proof:

$$\begin{aligned}(f^{-1}(\sigma))(x *_1 y) &= \sigma(f(x *_1 y))\\ &= \sigma(f(x) *_2 f(y)) \text{ since f is an epimorphism}\\ &\geq (\sigma(f(x) \wedge \sigma(f(y))) \text{ since } \sigma \text{ is a L-fuzzy BP – sub algebra}\\ &= (f^{-1}(\sigma))(x) \wedge f^{-1}(\sigma))(x))) \ \forall x, y \in X\end{aligned}$$

Thus $f^{-1}(\sigma)$ is a L-fuzzy BP- subalgebra of X_1.

Lemma 3.19 *An epimorphic image of a L- fuzzy BP- subalgebra satisfying sup property is a L- fuzzy BP- sub algebra. That is, let $f: X_1 \to X_2$ be an epimorphism of BP- algebras. If* μ is a L-fuzzy BP – subalgebra of X_1 with sup property, then f(μ) is a L fuzzy BP – subalgebra of X_2.

Proof:

Let *$f(x), f(y) \in f(X_1)$ and let $x_0 \in f^{-1}(f(x))$, and $y_0 \in f^{-1}(f(y))$, be such that*

$$\mu(x_0) = Sup_{a \in f^{-1}(f(x))} \mu(a)$$

$$\mu(y_0) = Sup_{b \in f^{-1}(f(y))} \mu(b) (f(\mu)(x*y)$$

$$= Sup_{a \in f^{-1}(f(x*y))} \mu(a) \text{ if } f^{-1}(x* y) \neq \emptyset.$$

Let $A = f^{-1}(f(x))$, $B = f^{-1}(f(y))$, $C = f^{-1}(f(x). f(y))$

$A * B = \{ x \in X_1 : x = a * b : a \in A, b \in B \}, x \in A * B$

$f(x) = f(a * b) = f(a) * f(b), x \in (f^{-1} f(a) * f(b))$ *implies* $A * B \subseteq C$.

Now,

$f(\mu)(f(a) *_2 f(b)) = \sup_{a \in f^{-1}(f(a) *_2 f(b)))} \mu(x)$

$= \sup_{x \in C} \mu(x) \geq \sup_{x \in A*B} \mu(x) \geq \sup_{a \in A, b \in B} \mu(a *_1 b)$

$\geq \sup_{a \in A, b \in B} (\mu(a) \wedge \mu(b))$

$= \sup_{a \in A, b \in B} (\mu(a) \wedge \mu(b))$

$= \sup_{a \in f^{-1}(f(x))} \mu(a) \wedge \sup_{b \in f^{-1}(f(x))} \mu(b)$

$= f(\mu(a)) \wedge f(\mu(a))$

Thus an epimorphic image of a L- fuzzy BP – subalgebra satisfying the sup property is a L-fuzzy BP – subalgebra.

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