On the Distribution of the Zeros of Lacunary type Polynomials

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ABSTRACT

In this paper we use matrix methods and Gereshgorian disk Theorem to present some interesting generalizations of some well known results concerning the distribution of the zeros of polynomial. Our results include as a special case some results due to A. Aziz and a result of Simon Reich-Lossar.

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Introduction and Statement of Results

The following result due to Cauchy [4] is well known in the theory of the distribution of the zeros of a polynomial.

Theorem A. Let

\[ P(z) = z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0, \]

be a polynomial of degree \( n \) then all the zeros of \( P(z) \) lie in the disk

\[ |z| < 1 + A. \]  (1)

where \( A = \max |a_j|, j = 0,1,2,\ldots,n-1. \)

About forty years ago, in connection with Cauchy’s Classical result (Theorem A) Simon Reich proposed and among others Lossers[6] verified that if \( a_{n-1} = 0, Q > 1 \), then all the zeros of

\[ P(z) = z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0, \]

lie in the circle

\[ |z| \leq Q + Q^2 + \cdots + Q^{n-1} \]  (2)

Aziz [2] generalized the problem to lacunary polynomials and showed that the assertion (2), remains valid even if we do not assume that \( Q > 1 \). In fact he proved:

Theorem B. Let

\[ P(z) = a_n z^n + a_r z^r + \ldots + a_1 z + a_0, \]

\( a_r \neq 0, 0 < r \leq n-1 \) be a polynomial of degree \( n \geq 2 \), with real or complex coefficients if
we prove:

**Theorem 1.** Let

\[ P(z) = a_n z^n + a_{r-1} z^{r-1} + \ldots + a_1 z + a_0, \]

where \( a_r \neq 0 \) \( 0 \leq r \leq n-1 \) be a polynomial of degree \( n \geq 2 \), with real or complex coefficients if \( t \) is any given positive number and

\[ Q = \left\{ \text{Max}_{0 \leq j \leq r} \left| \frac{a_j}{a_n} \right| \right\}^{\frac{1}{n}} \]

then all the zeros of \( P(z) \) lie in the disk

\[ |z| \leq Q + Q^2 + \ldots + Q^{r+1} \quad (3) \]

Where \( 0 \leq r \leq n-1 \). Other results of similar type were obtained among others by Alzer [1], Bell[3], Guggenheimer[5], Mohammad[7], Rahman[8], Walsh [10] (see also [9]).

As a generalization of Theorem B, we prove:

**Theorem 1.** Let

\[ P(z) = a_n z^n + a_{r-1} z^{r-1} + \ldots + a_1 z + a_0, \]

where \( a_r \neq 0 \) \( 0 \leq r \leq n-1 \) be a polynomial of degree \( n \geq 2 \), with real or complex coefficients if \( t \) is any given positive number and

\[ Q = \left\{ \text{Max}_{0 \leq j \leq r} \left| \frac{a_j}{a_n} \right| \right\}^{\frac{1}{n}} \quad (4) \]

then all the zeros of \( P(z) \) lie in the disk

\[ |z| \leq \frac{1}{t} \left\{ Q + Q^2 + \ldots + Q^{r+1} \right\} \quad (5) \]

where \( 0 \leq r \leq n-1 \).

Taking \( t = 1 \), in equation (5), this reduces to Theorem B.

We next present the following result which provides an interesting refinement of Theorem 1.

**Theorem 2.** Let

\[ P(z) = a_n z^n + a_{r-1} z^{r-1} + \ldots + a_1 z + a_0 \]

where \( a_r \neq 0 \) \( 0 \leq r \leq n-1 \) be a polynomial of degree \( n \geq 2 \), with real or complex coefficients if \( t \) is any given positive number and

\[ Q = \left\{ \text{Max}_{0 \leq j \leq r} \left| \frac{a_j}{a_n} \right| \right\}^{\frac{1}{n}} \]
then all the zeros of $P(z)$ lie in the disk

$$|z| \leq \frac{1}{t} \left\{ Q_t + \text{Max}(Q_t^2, Q_t^{r+1}) \right\}$$  \hspace{1cm} (6)$$

where $1 \leq r \leq n-1$. The following result immediately follows from Theorem 2 by taking $t = 1$:

**Corollary 1.** Let $P(z) = a_n z^n + a_{r-1} z^{r-1} + \ldots + a_1 z + a_0$

$a_r \neq 0 \ 0 \leq r \leq n-1$ be a polynomial of degree $n \geq 2$, with real or complex coefficients if $t$ is any given positive number and

$$Q_t = \left\{ \text{Max}_{0 \leq j \leq r} \left| \frac{a_j}{a_n} \right| \right\}^{\frac{1}{n}}$$

then all the zeros of $P(z)$ lie in the disk

$$|z| \leq Q + \text{Max}\{Q^2 + \ldots + Q^{r+1}\}$$  \hspace{1cm} (5)$$

where $1 \leq r \leq n-1$,

**Proof of the Theorems**

**Proof of Theorem 1.** The companion matrix of the polynomial

$$P(z) = a_n z^n + a_{r-1} z^{r-1} + \ldots + a_1 z + a_0$$

$a_r \neq 0 \ 0 \leq r \leq n-1$ of degree $n$ is

$$C = \begin{pmatrix}
0 & 0 & \ldots & 0 & \ldots & 0 & -a_0 t^{-1} \\
0 & 0 & \ldots & 0 & \ldots & 0 & a_n Q^{-1} \\
Q_t & 0 & \ldots & 0 & \ldots & 0 & -a_1 t^{-2} \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & a_n Q^{-2} \\
0 & 0 & \ldots & Q_t & \ldots & 0 & -a_r t^{-r-1} \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & a_n Q^{-r-1} \\
0 & 0 & \ldots & 0 & \ldots & Q_t & 0 \\
\end{pmatrix}$$
By hypothesis,

\[
Q_i = \left\{ \text{Max}_{0 \leq j \leq r} \left\{ \frac{a_j}{a_n} t^{n-j} \right\} \right\}^{\frac{1}{n}}
\]

therefore,

\[
\left| \frac{a_j}{a_n} t^{n-j} \right| \leq Q_i^n \quad \text{for } j = 0, 1, 2, \ldots, r \text{ and } Q_i \neq 0.
\]

(7)

We take the matrix

\[
P = \text{diag} \left\{ \left( \frac{Q_i}{t} \right)^{n-1}, \left( \frac{Q_i}{t} \right)^{n-2}, \ldots, \left( \frac{Q_i}{t} \right), 1 \right\}
\]

and form the matrix

\[
P^{-1}CP = \begin{pmatrix}
0 & 0 & \ldots & 0 & \ldots & 0 & -a_0 t^{n-1} \\
& Q_i & 0 & \ldots & 0 & \ldots & 0 & -a_1 t^{n-2} \\
& & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
& & & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \ldots & \frac{Q_i}{t} & \ddots & 0 & \frac{a_t t^{n-r-1}}{a_n Q_i^{n-r-1}} & \ddots \\
& & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \ldots & 0 & \ldots & \frac{Q_i}{t} & 0 & \ddots \\
& & & & & & & \ddots
\end{pmatrix}
\]

Applying Gereshgorian Theorem to the columns of \( P^{-1} CP \) and noting (7), it follows that all the eigen values of the matrix \( P^{-1} CP \) lie in the circle

\[
| z | \leq \text{Max} \left\{ \frac{Q_i}{t}, \sum_{j=0}^{r} \frac{a_j}{a_n} t^{n-j-1} \right\}
\]

\[
\leq \frac{1}{t} \text{Max} \left\{ \sum_{j=0}^{r} Q_i^{j+1} \right\}
\]

\[
= \frac{1}{t} \left\{ Q_i + Q_i^2 + \ldots + Q_i^{r+1} \right\}
\]
Since the matrix $P^{-1} CP$ is similar to the matrix $C$ and the eigen values of $C$ are the zeros of the polynomial $P(z)$, it follows that all the zeros of $P(z)$ lie in the circle

$$|z| \leq \frac{1}{t} \left(Q_r + Q_r^2 + \ldots + Q_r^{r+1}\right)$$

Which completes the proof of Theorem 1.

**Proof of Theorem 2.** The companion matrix of the polynomial

$$P(z) = a_n z^n + a_{r-1} z^{r-1} + \ldots + a_1 z + a_0$$

$a_r \neq 0$ for $0 \leq r \leq n-1$ of degree $n$ is given by

$$C = \begin{bmatrix}
0 & 0 & \ldots & 0 & \ldots & 0 & -\frac{a_r t^{r-1}}{a_n Q_t^{n-1}} \\
\frac{Q_t}{t} & 0 & \ldots & 0 & \ldots & 0 & -\frac{a_r t^{r-2}}{a_n Q_t^{n-1}} \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & \ldots & \frac{Q_t}{t} & \ldots & 0 & -\frac{a_r t^{r-1}}{a_n Q_t^{n-1}} \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & \ldots & 0 & \ldots & \frac{Q_t}{t} & 0
\end{bmatrix}$$

Proceeding similarly as in the proof of Theorem 1 and noting that

$$P = \text{diag} \left\{ \left(\frac{Q_t}{t}\right)^{n-1}, \left(\frac{Q_t}{t}\right)^{n-2}, \ldots, \left(\frac{Q_t}{t}\right), 1 \right\}$$

with

$$Q_t = \max_{0 \leq j \leq r} \left\{ \left| \frac{a_j}{a_n} t^{n-j} \right| \right\}^{\frac{1}{n}}$$
It follows that the matrix

\[
P^{-1}CP = \begin{pmatrix}
0 & 0 & \ldots & 0 & 0 & \frac{-a_0 t^{n-1}}{a_n Q_t^{n-1}} \\
\frac{Q_t}{t} & 0 & \ldots & 0 & 0 & \frac{-a_1 t^{n-2}}{a_n Q_t^{n-1}} \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & \ldots & \frac{Q_t}{t} & 0 & \frac{-a_r t^{n-r-1}}{a_n Q_t^{n-r-1}} \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & \ldots & 0 & \frac{Q_t}{t} & 0
\end{pmatrix}
\]

Applying Gereshgorian Theorem to the columns of \( P^{-1} CP \) and noting (7), it follows that all the eigen values of the matrix \( P^{-1} CP \) therefore that of \( C \) lie in the circle

\[
|z| \leq \max_{1 \leq j \leq r} \left\{ \frac{a_0 t^{n-1}}{a_n Q_t^{n-1}}, \frac{Q_t}{t}, \frac{a_1 t^{n-2}}{a_n Q_t^{n-1}}, \ldots, \frac{a_r t^{n-r-1}}{a_n Q_t^{n-r-1}}, \frac{Q_t}{t} \right\}
\]

Since the matrix \( P^{-1} CP \) is similar to the matrix \( C \) and the eigen values of \( C \) are the zeros of the polynomial \( P(z) \), therefore we conclude that all the zeros of \( P(z) \) lie in the circle denoted by (4). This proves Theorem 2 completely.

References


