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Pth Moment Exponential Stability of Impulsive Stochastic Differential Equations with Unbounded Delays

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ABSTRACT

Stochastic impulsive differential equations with Markovian switching have been applied extensively in various areas including ecology systems, neural networks, and control systems, and stability analysis is one fundamental premise of their applications. For two categories of Markovian switched impulsive stochastic differential functional equations with unbounded delays, this paper investigates the pth moment exponential stability by adopting stochastic Lyapunov stability theory and stochastic analysis approach. Several criteria on pth moment exponential stability have been acquired. In the proposed model, the time-varying coefficients and the hybrid impulsive effects are considered simultaneously. It can be seen that the criteria derived in this paper are more concise and the conditions are easier to verify compared with those existing results based on Razumikhin theory. Finally, two examples are illustrated to show the effectiveness of the theoretical findings.

Keywords: unbounded delay, impulsive effects, stochastic differential equations, Markovian switching; exponential stability.

1. Introduction

Stochastic differential equations with Markovian switching generated by a Markov chain are a special class of hybrid systems in which each subsystem switches randomly. In practice, abrupt changes will occur in their structure and parameters of many physical systems such as biological systems, aircraft systems, energy systems, and neural network systems, and stochastic differential equations with Markovian switching can model these systems commendably (Mao & Yuan, 2006; Yin & Zhu, 2010). On the other hand, impulses depict the phenomenon related to the variations of the system states at some discrete instantaneous moments. Besides, impulsive control can also be perceived as an important discontinuous control approach of nonlinear dynamical systems. Time delays are ubiquitous in the real world and can greatly affect the dynamic behaviour of the system, leading to instability or oscillations. At present, impulsive stochastic differential equations with time delays have aroused a great deal of interest from scholars and fruitful results have been obtained. By utilizing the Razumikhin method, pth moment stability of impulsive stochastic differential equations with bounded time delays is investigated in the literature (Wu et al., 2013; Hu et al., 2019), and the theory is further extended to autonomous impulsive stochastic systems with Markov switching (Gao et al., 2018). Based on the stochastic Lyapunov stability theory, the moment stability of a class of stochastic systems with bounded time delays and hybrid impulses has been investigated in the literature (Tran & Yin, 2023). Obviously, the Lyapunov stability method is more straightforward and effective compared with the Razumikhin technique. Meanwhile, systems with unbounded time delays have also been received wide attention, which includes infinite distributed time delays and unbounded time-varying time delays. For autonomous stochastic systems with infinite distributed time delays, the literature (Wu & Hu, 2011; Pavlovic & Jankovic, 2012; Mei et al., 2021) studied the attraction set, moment stability with general decay rate and discrete feedback stabilization problem. Soon afterwards, the theory is generalized the stability analysis to functional systems with infinite distributed delays and time-varying coefficients (Li & Xu, 2021). For unbounded time-varying delay systems with impulsive effects, the moment stability has been analyzed in the literature (Xu & Zhu, 2021, 2022) in virtue of average dwell time method and Razumikhin technique.

It can be observed that the existing results mainly concentrate on the analysis of stochastic systems with time-varying delays and stability or instability impulses, but the cases where time-varying coefficients, unbounded delays and hybrid impulses are incorporated into the systems have not yet been considered. Additionally, the used method is the Razumikhin method, by which the Lyapunov function is easily constructed while the conditions are complicated to verify. Based on the aforementioned discussion, this paper introduces the decay function with impulsive effects, and investigates the stochastic hybrid impulsive differential equations with Markovian switching and unbounded time delays in virtue of Lyapunov stability theory and stochastic analysis technique. We also obtain several criteria on pth moment exponential stability of non-autonomous stochastic differential equations with hybrid impulses and two types of unbounded time delays, including infinitely distributed time delays and unbounded time-varying time delays. The method is more direct and effective, and the results can be seen as the extensions to the literature (Tran & Yin, 2023; Mei et al., 2021; Li & Xu, 2021; Xu & Zhu, 2021, 2022).

2. Preliminaries

Firstly, we will introduce the following notations. R denotes the set of real number, R_+ denotes the set of non-negative real numbers, R^n is an n-dimensional Euclidean space.
$w(t)=(w_1(t),w_2(t),\cdots,w_m(t))$ represents the $m-$dimensional Brownian motion defined in the probability space $(\Omega,F,\{F_t\}_{t\geq 0},P)$. $\delta(t),t\geq 0$ is a right-continuous Markov chain defined on the probability space taking the value $S=\{1,2,\ldots,M\}$ with the following transition probability.

$$P\{\delta(t+\Delta)=j\mid\delta(t)=i\}=\begin{cases}\delta_{ij}\Delta+o(\Delta),i\neq j,\\1+\delta_{ii}\Delta+o(\Delta),i=j,\end{cases}$$

where $\delta_{ij}>0,i\neq j$ and $\sum_{j=1}^{n}\delta_{ij}=0$. $PC^b((-\infty,0];R^n)$ represents the set of piecewise bounded continuous functions ϕ:$(-\infty,0]\to R^n$ with norm $\|\phi\|=\sup_{s\leq 0}|\phi(s)|$. $PC^b_{F_0}((-\infty,0];R^n)$ indicates F_0 measurable function set $PC((-\infty,0];R^n)$ satisfying $\sup_{s\leq 0}E|\phi(s)|^p$.

Consider the following impulsive stochastic functional differential equations with Markovian switching

$$\begin{cases}dx(t)=f(x_t,t,\delta(t))dt+f(x_t,t,\delta(t))dw(t),t\neq t_k,\\x(t_k)=I_k(x(t_k^-)),t=t_k,\end{cases}\tag{1}$$

where $x_t = x_t(\theta) = \{x(t+\theta), \theta \in (-\infty, 0]\}$, and $f : PC^b((-\infty,0]; R^n) \times R \times S \to R^n$, $g : PC^b((-\infty]; R^n) \times R \times S \to R^{n\times m}$ are two Borel measurable functions. $\zeta(\theta) = \zeta \in PC^b_{F_0}((-\infty,0]; R^n)$ denotes the initial value.

In order to derive the main results, we make the following necessary assumptions.

(H1) (Local Lipschitz condition) For $\forall h > 0$, there exists a constant $K_h > 0$ such that

$$|f(\phi_1,t,i) - f(\phi_2,t,i)| \vee |g(\phi_1,t,i) - g(\phi_2,t,i)| < K_h \|\phi_1 - \phi_2\|,$$

where ϕ_1, $\phi_2 \in PC^b((-\infty,0]; R^n)$, $\|\phi_1\| \vee \|\phi_2\| \le h$ and $(t,i) \in R_+ \times S$.

(Linear growth condition) For $\phi \in PC^b((-\infty,0]; R^n)$, there exists one positive constant K_0 such that

$$|f(\phi,\mathrm{t},\mathrm{i})| + |g(\phi,t,i)| \le K_0(1 + \|\phi\|).$$

(H2) Suppose that there exist two positive constants c_1, c_2 and a sequence of positive constants $\{\mu_k\}_{k\in N}$ such that

$$c_1|x|^p \le V(x,i) \le c_2|x|^p,\ V(I_k(x),i) \le \mu_k V(x,i).$$

(H3) There exists some probability measures $\alpha_{\mathrm{j}}(s)$, $j = 1,2,\cdots,q$ defined on $(-\infty,0]$ and a series of continuous bounded functions $\beta_0(t)$, $\beta_j(t)$, $j = 1,2,\cdots,q$ such that

$$LV(\phi,\mathrm{i}) \le \beta_0(t)V(\phi(0),i) + \sum_{j=1}^{q} \beta_j(t) \int_{-\infty}^{0} V(\phi(\theta),i)\, d\alpha_j(\theta),$$

where $(\phi,\ \mathrm{i}) \in PC^b((-\infty,0]; R^n) \times R_+ \times S$.

In addition, we introduce the following function:

$$\varphi(\lambda(t),t) = \int_0^t \lambda(s)ds + \sum_{t_m \le t} In\mu_m.$$

Remark 1. Under the standard hypothesis (H1), there exists a unique solution to the unbounded time-delay system $dx(t) = f(x_t,t,\delta(t)) + g(x_t,t,\delta(t))dw(t)$ and the solution keeps pth moment boundedness in the time interval $[t_0, t_1)$. Combining with the definition of impulse, it is derived that the system (1) also has a unique solution and it keeps pth moment boundedness in the time interval $[t_1, t_2)$. By repeating the inducing, it follows that the system (1) has a unique solution and it maintains pth moment boundedness in any time interval $[t_k, t_{k+1})$. On the other hand, the introduced function $\varphi(\lambda(t),t)$ can be viewed as an extension of the function $\varphi(c,t)$ in the literature [6], and it is applicable to the stability analysis of non-autonomous systems.

3. Main Results

In this section, by stochastic Lyapunov approach, some inequality techniques and stochastic analysis theory, we will discuss the pth moment stability of non-autonomous stochastic systems with unbounded delays and hybrid impulse.

Theorem 1. Let Assumptions (H1), (H2) and (H3) hold. if there exist a continuous bounded function $\eta_1(t)$ and two constants $\rho_1 \le 0$, $\sigma_1 \ge 0$ such that

$$\psi(\theta) = -\int_{t+\theta}^{t} \eta_1(u)du - \sum_{t+\theta<t_m \le t} In\mu_m \le \rho_1\theta + \sigma_1, t \ge 0, -\infty < \theta \le 0,$$

$$\beta_0(t) + \frac{c_2}{c_1}\sum_{j=1}^{q}\beta_j(t)\int_{-\infty}^{0} e^{\rho_1\theta+\sigma_1} d\alpha_j(\theta) \le \eta_1(t), t \ge 0,$$

where $\eta_1(t)$ satisfies $\eta_1(t) = 0, t < 0,$ then we have that

$$E|x(t)|^p \le \tilde{K}E\|\zeta\|^p e^{\varphi(\eta_1(t),t)},$$

where $\tilde{K} = K'E\|\zeta\|^p$ and K' denotes one sufficient large positive constant. In particular, if there exist constants $\rho_2 > 0$, $\sigma_2 \ge 0$ such that

$$\varphi(\eta_1(t),t) = \int_0^t \eta_1(s)ds + \sum_{t_m \le t} In\mu_m \le -\rho_2 t + \sigma_2,$$

then the system (1) is exponentially stable in pth moment.

Proof According to Assumptions (H1) and (H2), together with Remark 1, we can easily obtain that the unique global solution of the equation exists and satisfies $E|x(t)|^p < +\infty$, $t \in R$. Moreover, by definition of $\varphi(.,.)$, we infer that $\inf_{t\in[0,t_1)} e^{\varphi(\eta_1(t),t)} > 0$. Accordingly, there is always a sufficiently large positive constant K' such that $EV(x(t),\delta(t)) < \tilde{K}e^{\varphi(\eta_1(t),t)}$, $t \in [0,t_1)$, where $K' > 0$ and $\tilde{K} = K'E\|\zeta\|^p$. Subsequently, we will claim that

$$EV(x(t),\delta(t)) < \tilde{K}e^{\varphi(\eta_1(t),t)}, \quad t \in [0,t_n). \tag{2}$$

In the case of $n = 1$, it is obvious that equation (2) holds. Assume that equation (2) is satisfied for $n = 2,3,\cdots,k$. Moreover, we prove that it holds for $n = k+1$. For simplicity, define the function $\Phi(t) = \tilde{K}e^{\varphi(\eta_1(t),t)}$. Noting that $EV(x(t),\delta(t)) < \Phi(t)$, $t \in [0,t_k)$, we only need to prove that $EV(x(t),\delta(t)) < \Phi(t)$, $t \in [t_k,t_{k+1})$.

When $t = t_k$, we calculate that

$$EV(x(t_k),\delta(t_k)) \le \mu_k EV(x(t_k^-),\delta(t_k^-)) < \mu_k\tilde{K}e^{\varphi(\eta_1(t_k^-),t_k^-)} \le \mu_k\tilde{K}e^{\int_0^{t_k}\eta_1(s)ds + \sum_{t_m \le t_k^-} In\mu_k}$$

$$\le \tilde{K}e^{\int_0^{t_k}\eta_1(s)ds+\sum_{t_m\le t_k} In\mu_k} = \Phi(t_k),$$

which means $EV(x(t_k),\delta(t_k)) < \Phi(t_k)$.

If assertion (2) does not hold for $n = k+1$, then there exists $t^* \in (t_k, t_{k+1})$ such that

$$EV(x(t),\delta(t)) < \Phi(t),\ t \in [0,t^*) \text{ and } EV(x(t^*),\delta(t^*)) = \Phi(t^*). \quad (3)$$

Choose a sufficiently large positive number λ such that $\lambda t + \int_0^t \eta_1(s)ds \ne 0, t \ge 0$. According to the $It\hat{o}$ formula, we have

$$e^{\lambda t^*}EV(x(t^*),\delta(t^*)) = e^{\lambda t_k}EV(x(t_k),\delta(t_k)) + E\int_{t_k}^{t^*} e^{\lambda s}[\lambda V(x(s),\delta(s)) + LV(x(s),\delta(s))]ds$$

$$< e^{\lambda t_k}\tilde{K}e^{\varphi(\eta_1(t_k),t_k)} + \int_{t_k}^{t^*} e^{\lambda s}\left[(\lambda+\beta_0(s))EV(x(s),\delta(s)) + \sum_{j=1}^{q}\beta_j(s)\int_{-\infty}^{0} EV(x(s+\theta),\delta(s))d\alpha_j(\theta)\right]ds$$

$$\le e^{\lambda t_k}EV(x(t_k),\delta(t_k))$$

$$+\int_{t_k}^{t^*} e^{\lambda s}\left[(\lambda+\beta_0(s))EV(x(s),\delta(s)) + \sum_{j=1}^{q}\beta_j(s)\frac{c_2}{c_1}\int_{-\infty}^{0} EV(x(s+\theta),\delta(s+\theta))d\alpha_j(\theta)\right]ds,$$

By condition (3), we have

$$EV(x(s),\delta(s)) < \Phi(s) = \tilde{K}e^{\varphi(\eta_1(s),s)}, s \in [t_k, t^*),$$

$$EV(x(s+\theta),\delta(s+\theta)) < \Phi(s) = \tilde{K}e^{\varphi(\eta_1(s+\theta),s+\theta)}$$

$$\le \tilde{K}e^{\varphi(\eta_1(s),s)}e^{\varphi(\eta_1(s+\theta),s+\theta)-\varphi(\eta_1(s),s)}$$

$$\le \tilde{K}e^{\varphi(\eta_1(s),s)}e^{-\int_{s+\theta}^{s}\eta_1(s)ds-\sum_{s+\theta<t_m\le s} In\mu_m}$$

$$\le \tilde{K}e^{\varphi(\eta_1(s),s)}e^{\rho_1\theta+\sigma_1}, \quad s \in [t_k, t^*), \theta \in (-\infty, 0). \quad (4)$$

Substituting inequality (4) into the previous inequality, we have that

$$e^{\lambda t^*}EV(x(t^*),\delta(t^*)) < \tilde{K}e^{\lambda t_k+\varphi(\eta_1(t_k),t_k)} + \tilde{K}\int_{t_k}^{t^*} e^{\lambda s+\varphi(\eta_1(s),s)}\left[\lambda+\beta_0(s)+\sum_{j=1}^{q}\beta_j(s)\int_{-\infty}^{0} e^{\rho_1\theta+\sigma_1}d\alpha_j(\theta)\right]ds$$

$$\le \tilde{K}e^{\lambda t_k+\varphi(\eta_1(t_k),t_k)} + \tilde{K}\int_{t_k}^{t^*} e^{\lambda s+\varphi(\eta_1(s),s)}[\lambda+\eta_1(s)]ds$$

$$\le \tilde{K}e^{\lambda t_k+\varphi(\eta_1(t_k),t_k)} + \tilde{K}e^{\sum_{t_m\le t_k} In\mu_m}\int_{t_k}^{t^*} e^{\lambda s+\int_0^s \eta_1(u)du}[\lambda+\eta_1(s)]ds$$

$$\leq \widetilde{K}e^{\lambda t_k+\varphi(\eta_1(t_k),t_k)}+\widetilde{K}e^{\sum_{t_m\leq t_k} In\mu_m}\left[e^{\lambda t^*+\int_0^{t^*}\eta_1(u)du}-e^{\lambda t_k+\int_0^{t_k}\eta_1(u)du}\right]$$

$$\leq \widetilde{K}e^{\lambda t_k+\varphi(\eta_1(t_k),t_k)}+\widetilde{K}e^{\varphi(\eta_1(t^*),t^*)+\lambda t^*}-\widetilde{K}e^{\lambda t_k+\varphi(\eta_1(t_k),t_k)}$$

$$\leq \widetilde{K}e^{\varphi(\eta_1(t^*),t^*)+\lambda t^*}.$$

which leads to a contradiction $EV(x(t^*),\delta(t^*))<\widetilde{K}e^{\varphi(\eta_1(t^*),t*)}=\Phi(t^*)$. Hence, it holds for $n=k+1$, i.e. $EV(x(t),\delta(t))<\widetilde{K}e^{\varphi(\eta_1(t),t)}$, $t\geq 0$.

In particular, when $\varphi(\eta_1(t),t)\leq -\rho_2 t+\sigma_2$, then $\lim_{t\to+\infty}\frac{E|x(t)|^p}{t}\leq -\rho_2$, which implies that system (1) is exponentially stable in pth moment.

Just now, we have tackled the pth exponential stability with the infinite distribution delay. Moreover, we will consider the case of unbounded time-varying delays. Let x_t be composed of $x(t-\tau_j(t))$, $j=1,2,\cdots,q$, where $\tau_j(t)$ stands for unbounded functions. Meanwhile, the following assumption is provided.

(H4) Let there exist a series of continuous bounded functions $\gamma_0(t),\gamma_1(t),\cdots,\gamma_q(t)$, such that

$$LV(\phi,i)\leq \gamma_0(t)V(\phi(0),i)+\sum_{j=1}^{q}\gamma_j(t)V(\phi(-\tau_j(t),i)),$$

where $(\phi,i)\in PC((-\infty,0];R^n)\times S$.

Theorem 2. Let Assumptions (H1), (H2) and (H4) hold. If there exist constants $\bar{\rho}_j\in R_+$, $\bar{\sigma}_j\in R_+$, $j=1,2,\ldots,q$ and one continuous bounded function $\eta_2(t)$ such that

$$-\int_{t-\tau_j(t)}^{t}\eta_2(s)ds-\sum_{t-\tau_j(t)<t_m\leq t}In\mu_m\leq \bar{\rho}_j\tau_j(t)+\bar{\sigma}_j=\psi_j(t),$$

$$\gamma_0(t)+\frac{c_2}{c_1}\sum_{j=1}^{q}\gamma_j(t)e^{\psi_j(t)}\leq \eta_2(t),$$

then we have $E|x(t)|^p\leq \widetilde{K}E\|\zeta\|^p e^{\varphi(\eta_2(t),t)}$. Furthermore, if there exist constants $\rho_0>0$, $\sigma_0\geq 0$ such that $\varphi(\eta_2(t),t)\leq -\rho_0 t+\sigma_0$, then the system is exponentially stable in pth moment.

Proof Noting that

$$EV(x(s-\tau_j(s)),\delta(s-\tau_j(s)))<\Phi(s)=\widetilde{K}e^{\varphi(\eta_2(s-\tau_j(s)),s-\tau_j(s))}$$

$$\leq \widetilde{K}e^{\varphi(\eta_2(s),s)}e^{\varphi(\eta_2(s-\tau_j(s)),s-\tau_j(s))-\varphi(\eta_2(s),s)}$$

$$\le \tilde{K}e^{\varphi(\eta_2(s),s)}e^{-\int_{t-\tau_j(t)}^{t}\eta_2(s)ds-\sum_{t-\tau_j(t)<t_m\le t} In\mu_m}$$

$$\le \tilde{K}e^{\varphi(\eta_2(s),s)}e^{\psi_j(s)},$$

Similar to the proof of Theorem 1, we can derive the assertion.

Remark 2. Recently, pth moment exponential stability of stochastic impulsive differential systems with bounded time delays and Markov switching has been investigated in the literature (Tran & Yin, 2023), and the theory is extended to non-autonomous impulsive systems with unbounded time delays in this paper. Besides, the moment stability of autonomous systems and non-autonomous systems has been investigated in the literature (Wu & Hu, 2011; Pavlovic & Jankovic, 2012; Mei et al., 2021; Li & Xu, 2021). Nevertheless, the moment stability of non-autonomous stochastic differential equations with unbounded delays and hybrid impulse has not been taken into account. In this paper, the decay function with impulse is introduced. Moreover, by stochastic analysis theory, some criteria on pth moment stability of stochastic systems with unbounded delays and hybrid impulse are given. Different from the Razumikhin method proposed in references (Xu & Zhu, 2021, 2022), stochastic Lyapunov stability theory can be employed directly. The conditions are easier to verify and the effects of hybrid impulse are incorporated.

4. Numerical example

In this section, we will give two specific numerical examples that demonstrate the validity of the results of the aforementioned analysis.

Example 1. Consider the following stochastic differential equation

$$\begin{cases} dx(t) = f(x(t), \int_{-\infty}^{0} x(t+\theta)d\alpha_1(\theta), t, \delta(t))dt + g(x(t), \int_{-\infty}^{0} x(t+\theta)d\alpha_1(\theta), t, \delta(t))dw(t), t \ne t_k, \\ x(t_k) = \varepsilon_k x(t_k^-), t = t_k. \end{cases}$$

where $\alpha_1(\theta) = e^{18\theta}$. We take $S = \{1,2\}$, $\Gamma = \begin{pmatrix} -0.8 & 0.8 \\ 1.2 & -1.2 \end{pmatrix}$, $z = \int_{-\infty}^{0} x(t+\theta)d\alpha_1(\theta)$ and set

$f(x,z,t,1) = (-1.9 - 1.5|\sin t|)x + 0.8z$, $f(x,z,t,2) = (-1.95 - 1.5|\sin t|)x + 0.9z$.

For the above random impulse system, let $V(x,1) = 0.8x^2$, $V(x,2) = x^2$, then we have

$$LV(x,1) = (-3.04 - 2.4|\sin t|)x^2 + 1.28xz + 0.64(x+0.6z)^2 + 0.16x^2$$
$$\le -(1.2 + 3|\sin t|) \times (0.8x^2) + 1.376 \times (0.8z^2),$$

$$LV(x,2) = (-3.9 - 3|\sin\theta|)x^2 + 1.8xz + (x+0.5z)^2 - 0.23x^2 \le -(1.24 + 3|\sin t|)x^2 + 1.4z^2.$$

By utilizing Holder's inequality, we have $z^2 = z = [\int_{-\infty}^{0} x(t+\theta)d\alpha_1(\theta)]^2 \le \int_{-\infty}^{0} x^2(t+\theta)d\alpha_1(\theta)$.

Hence, we get that $LV(x,i) \le -(1.2+3|\sin t|)V(x,t,i)+1.4\int_{-\infty}^{0} V(x(t+\theta),t+\theta,i)d\alpha_1(\theta)$. Take $\beta_0(t)=-1.2-3|\sin t|, \beta_1(t)=1.4, \quad \varepsilon_k=\dfrac{\sqrt{2}}{2}, t_k-t_{k-1}=0.25, k=1,2,\cdots, \eta_1(t)=3-3|\sin t|$, $\mu_m=0.5$. We calculate that

$$\psi(\theta)=-\int_{t+\theta}^{t}\eta_1(u)du-\sum_{t+\theta<t_m\le t} In\mu_m \le -(4In2)\theta+In2,$$

$$\beta_0(t)+\frac{c_2}{c_1}\beta_1(t)\int_{-\infty}^{0}e^{\rho_1\theta+\sigma_1}d\alpha_1(\theta)=-1.2-3|\sin t|+1.75\int_{-\infty}^{0}18e^{(18-4In2)\theta+In2}d\theta \le 3-3|\sin t|=\eta_1(t).$$

Since $\int_0^t(3-3|\sin t|)dt+\sum_{t_m\le t}\mu_m \le (3-\frac{6}{\pi}-4In2)t+In2+\pi \le -1.67t+In2+\pi$, by Theorem 1, it follows that the above system is exponentially stable in pth moment.

Example 2. Consider unbounded time delay systems with hybrid impulse.

$$\begin{cases} dx(t)= f(x(t),x(\frac{1}{2}t),t,\delta(t))dt+g(x(t),x(\frac{1}{2}t),t,\delta(t))dw(t), \ t\neq t_k, \\ x(t_k)=\varepsilon_k x(t_k^-), \ t=t_k, \end{cases}$$

where $f(x,z,t,1)=(-1.9-0.5|\sin t|)x+1.5z, \ f(x,z,t,2)=(-2-0.5|\sin t|)x+1.8z$.

We take $S=\{1,2\}, \ \Gamma=\begin{pmatrix}-2 & -2\\ 3 & -3\end{pmatrix}, z=x(\frac{1}{2}t)e^{-\frac{1}{2}\psi(t)}, \psi(t)=\frac{1}{2}(\frac{2}{\pi}-In\frac{3}{2})t+\pi-2+In2.$

For the above random impulse system, let $V(x,1)=V(x,2)=x^2$, then we have $LV(x,1)\le(-2-|\sin t|)x^2+1.5z^2, LV(x,2)\le(-1.84-|\sin\theta|)x^2+1.8z^2$.

which implies that $LV(x,i)\le(-1.8-|\sin t|)V(x(t),i)+1.8e^{-\psi(t)}V(x(0.5t),i)$.

Let $\gamma_0(t)=-1.8-|\sin t|, \ \gamma_1(t)=1.8e^{-\psi(t)}, \varepsilon_{2k-1}=\dfrac{\sqrt{2}}{2}, \varepsilon_{2k}=\sqrt{3}, t_k-t_{k-1}=0.5.$

By computing we have that

$$\mu_{2k-1}=\frac{1}{2}, \mu_{2k}=3, k=1,2,\cdots,$$

$$-\int_{0.5t}^{t}-|\sin u|du-\sum_{0.5t<t_m\le t}In\mu_m \le \frac{1}{2}(\frac{2}{\pi}-In\frac{3}{2})t+\pi-2+In2=\psi(t).$$

It can be further verified that $\gamma_0(t)+\dfrac{c_2}{c_1}\gamma_1(t)e^{\psi(t)}=-1.8-|\sin t|+1.8\le-|\sin t|=\eta_1(t)$.

Since $\int_0^t -|\sin t|dt + \sum_{t_m \le t} \mu_m \le (-\frac{2}{\pi} + In\frac{3}{2})t + In3 + \pi - 2 \le -0.2314t + In3 + \pi - 2$, by Theorem 1, it follows that the above system is exponentially stable in pth moment.

Remark 3. In Example 1, infinitely distributed time delay and stable impulses are considered. In Example 2, unbounded proportional delays and hybrid impulses are considered, where hybrid impulses contain both stable and unstable impulses and stable impulses are dominant. When the original non-autonomous system are stable, impulsive effects can reduce the exponential decay rate of the system.

5. Conclusion

The paper is concerned with pth moment exponential stability of two categories of unbounded time-delay stochastic differential equations with hybrid impulse and time-varying coefficients. Firstly, the decay function containing the hybrid impulse is introduced. Subsequently, pth moment exponential stability of stochastic differential equations is investigated by stochastic Lyapunov stability theory and stochastic analysis approach. Compared with the standard Razumikhin method, this method is more direct and effective. In future, our theory can be extended to stochastic differential systems with Levy noise and stochastic network systems.

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