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ON $(\in, \in \vee q)$-FUZZY PRIME BI-IDEALS OF NEAR RING

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ABSTRACT

In this paper using the idea of quasi coincidence of a fuzzy point with a fuzzy set, we introduce the notion of $(\in, \in \vee q)$-fuzzy prime bi-ideals and semiprime bi-ideals. Also we investigate some related properties of these fuzzy substructures.

Key words: Near-ring, Fuzzy point, Quasi-coincidence, $(\in, \in \vee q)$-fuzzy prime bi-ideals, semi-prime bi-ideals.

1. Introduction

In 1965, L. A. Zadeh [1] introduced the concept of fuzzy set. Using this concept, many authors generalize several notions of algebra. In 1971 Rosenfield [2] defined fuzzy subgroups and gave some of its properties. Since then, the study of fuzzy algebraic structures has been pursued in many directions such as groups, modules, vector space and so on. In1982 Wang-Jin Liu [3] introduced the notion of fuzzy sub-ring and ideals. Subsequently, Mukharjee and Sen [4] Swamy and Swamy [5], Yue [6] Dixit et al [7], Raj Kumar and [8], Zie [9] developed the theory of fuzzy rings. Since then many researchers explored on the generalization of the notions of fuzzy set and its application to many mathematical branches. S. Abou zaid[10] introduced the notion of a fuzzy sub near-ring and fuzzy ideals of near-ring. The concept of the quasi-coincidence of a fuzzy point with a fuzzy subset was introduced by Pu Pao-ming and Liu Ying-ming [11] in 1980. The concept of "belongingness" and "quasi-coincidence" of a fuzzy point with a fuzzy set played a vital role to generate some different types of fuzzy subgroups. In [12,13] Bhakat and Das have introduced the notion of $(\in, \in \vee q)$-fuzzy sub-groups and notion of $(\in, \in \vee q)$-fuzzy sub-ring. Davvaz [14] extended these results to near-ring. In [15] Kazanci and Yamak study $(\in, \in \vee q)$-fuzzy bi ideals of a semi group. Dheena and Coumaressane [16] introduced the notion of an $(\in, \in \vee q_k)$-fuzzy quasi-ideals and bi-ideals of near ring. Bashir [17] introduced the notion of prime bi-ideal and strongly prime fuzzy bi-ideals in near-ring. In this paper using the idea of quasi coincidence of a fuzzy point with a fuzzy set, we introduce the notion of $(\in, \in \vee q)$-fuzzy prime bi-ideals and semiprime bi-ideals. Also we investigate some related properties of these fuzzy substructures.

2. Preliminaries

In this section we will briefly recall some basic notion.

Definition 2.1. A near-ring N is a system with two binary operations + and . such that :

(i) (N , +) is a group,

(ii) (N , .) is a semi group,

(iii) $(x + y)z = xz + yz$ for all $x, y, z \in N$.

In a near-ring only one distributive law holds (left or right).

Definition 2.2. Let N be a near-ring. A subset I of N is said to be an ideal of N if

(i) (I , +) is a normal subgroup of N,

(ii) $IN \subseteq I$

(iii) $x(y + z) - xy \in I$, for all $z \in I$ and $x, y \in N$.

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If I satisfies (i) and (ii), then I is called a right ideal of N. If I satisfies (i) and (iii), then I is called left ideal of N.

Definition 2.3. An ideal P of N is called prime if for all ideals I , J of N such that $IJ \subseteq P$ implies either $I \subseteq P$ or $J \subseteq P$.

Definition 2.4. An ideal I of N is semiprime if and only if for all ideals J of N such that $J^2 \subseteq I$ implies $J \subseteq I$.

Definition 2.5. Let N be a near-ring. For any two subsets A and B of N define an operation '$*$' is given by $A * B = \{ x(y + z) - xy \mid x, y \in A, z \in B\}$.

Definition 2.6. A subgroup B of (N , +) is said to be a bi-ideal of N if $BNB \cap (BN) * B \subseteq B$.

Definition 2.7. A bi-ideal B of a near-ring N is called a prime bi-ideal of N if $XY \subseteq B$ implies $X \subseteq B$ or $Y \subseteq B$ for any bi-ideals X ,Y of N.

Definition 2.8. A bi-ideal B of a near-ring N is called a semiprime bi-ideal of N if $X^2 \subseteq B$ implies $X \subseteq B$ for any bi-ideal X of N.

Definition 2.9. Let X be any non-empty set. A mapping $\mu : X \rightarrow [0,1]$ is called a fuzzy subset of X .

Definition 2.10. Let $A \subseteq X$. Then the function $\mu : X \rightarrow [0,1]$ defined by

$$\mu(x) = 1 \quad \text{when } x \in A \text{ and}$$
$$= 0 \quad \text{otherwise}$$

is a fuzzy subset of X, which is the characteristics function χ_A of A.

Definition 2.11. Let X be a any non empty set. A fuzzy subset μ in X defined by

$$\mu(y) = t \quad (\neq 0) \quad \text{if } y = x$$
$$= 0 \quad \text{otherwise}$$

is said to be a fuzzy point with support x and value t and is denoted by x_t.

Definition 2.12. If x_t is a fuzzy point of X, where $x \in X$ and $t \in (0,1]$, and $a \in N$ then $ax_t = (ax)_t$.

Definition 2.13. A fuzzy point x_t is said to belong to (resp. be quasi coincident with) a fuzzy set μ, written us $x_t \in \mu$ (resp. $x_t q \mu$) if $\mu(x) \geq t$ or (resp. $\mu(x) + t > 1$) :

if $\mu(x) \geq t$ or $\mu(x) + t > 1$ then we write $x_t \in \vee q \mu$.

Definition 2.14. Let X be any non empty set and μ is a fuzzy subset of X. For all $t \in (0,1]$, Denote the set $\mu_t = \{ x \in X \mid \mu(x) \geq t \}$. Then μ_t is called level subset of μ.

Definition 2.15. Let μ and δ be any two fuzzy subset of N. Then $\mu \cap \delta$, $\mu * \delta$ and $\mu\delta$ is defined by

$$(\mu \cap \delta)(x) = \min\{\mu(x), \delta(y)\}$$

$$(\mu * \delta)(x) = \begin{cases} \sup\limits_{x=a(b+c)-ab} \{ \min\{\mu(a), \delta(c)\}\}, & \text{If x is expressible as } x=a(b+c)-ab \\ = 0 & \text{otherwise} \end{cases}$$

$$(\mu\delta)(x) = \begin{cases} \bigvee\limits_{x=yz} \{\min\{\mu(y), \delta(z)\}\} & \text{If x is expressible as } x = yz. \\ = 0 & \text{otherwise. for all } x, y, z, a, b, c \in N. \end{cases}$$

Clearly for any fuzzy points x_t and y_r in N, we have $x_t y_r = (xy)_{t \wedge r}$.

Definition 2.16. A fuzzy subset μ of a near-ring is called a fuzzy bi-ideal of N if

(i) $\mu(x - y) \geq \min\{\mu(x), \mu(y)\}$,

(ii) $\mu(xyz) \geq \min\{\mu(x), \mu(z)\}$, for all $x, y, z \in N$.

A fuzzy subgroup μ of N is called a fuzzy bi-ideal of N if $\mu > (\mu N \mu) \cap (\mu N * \mu)$.

Definition 2.17. An $(\in, \in \vee q)$-fuzzy subgroup μ of N is called an $(\in, \in \vee q)$-fuzzy bi-ideal of N if $\mu(x) \geq \min\{((\mu N \mu) \cap (\mu N * \mu))(x), 0.5\}$, for all $x \in N$.

Remarks 2.18. Every fuzzy bi-ideal of N is an$(\in, \in \vee q)$-fuzzy bi-ideal of N.

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Remarks 2.19. Let μ and δ be any two $(\in, \in \vee q)$-fuzzy bi-ideal of N. Then $\mu \cap \delta$ is also an $(\in, \in \vee q)$-fuzzy bi-ideal of N.

Definition 2.20. If μ is a fuzzy subset of N, then we denote $\mu_* = \{x \in N \mid \mu(x) = \mu(0)\}$.

Definition 2.21. A fuzzy subset μ of N is said to be an $(\in, \in \vee q)$-fuzzy ideal of N if for all x, y, $z \in N$ and for all r, $t \in (0. 1]$

(1) $x_r . y_t \in \mu$ implies $(x - y)_{\min(r,t)} \in \vee q\ \mu$

(2) $x_t \in \mu$ and $y \in N$ implies $(y + x - y)_t \in \vee q\ \mu$

(3) $x_t \in \mu$ and $y \in N$ implies $(x y)_t \in \vee q\ \mu$

(4) $z_t \in \mu$ and $x,y \in N$ implies $(x(y + z) - xy)_t \in \vee q\ \mu$

3. $(\in, \in \vee q)$-fuzzy prime bi-ideals.

Definition 3.1. Let μ be an $(\in, \in \vee q)$-fuzzy bi-ideal of N. Then μ is said to be

(i) prime bi-ideal, if for all x, $y \in N$ and r, $t \in (0,1]$

$$x_r y_t \in \mu \text{ implies } x_r \in \vee q \mu \text{ or } y_t \in \vee q \mu$$

(ii) semiprime bi-ideal if for all $x \in N$ and $r \in (0,1]$

$$x_r^{\ 2} \in \mu \text{ implies } x_r \in \vee q \mu .$$

Theorem 3.2. Let μ be an $(\in, \in \vee q)$-fuzzy prime bi-ideal of N. Then the following statement are hold :

$\mu(x) \vee \mu(y) \geq \min(\mu(xy), 0.5)$, for all x, $y \in N$.

Proof : Let μ is an $(\in, \in \vee q)$-fuzzy prime bi-ideal of N. If possible let x, $y \in N$ be such that $\mu(x) \vee \mu(y) < \min(\mu(xy), 0.5)$. Choose r such that $\mu(x) \vee \mu(y) < r < \min(\mu(xy), 0.5)$. Then $(xy)_r \in \mu$ but $x_r \overline{\in \vee q}\ \mu$ and $y_r \overline{\in \vee q}\ \mu$, a contradiction. Thus $\mu(x) \vee \mu(y) \geq \min(\mu(xy), 0.5)$, for all x, $y \in N$.

Theorem 3.3. Let μ be an $(\in,\in\vee q)$-fuzzy semiprime bi-ideal of N if and only if $\mu(x) \geq \min(\mu(x^2), 0.5)$, for all $x \in N$.

Proof : Let μ is an $(\in,\in\vee q)$-fuzzy semiprime bi-ideal of N. If possible let $x \in N$ be such that $\mu(x) < \min(\mu(x^2), 0.5)$. Choose r such that $\mu(x) < r < \min(\mu(x^2), 0.5)$. .Then $(x^2)_r \in \mu$ but $x_r \overline{\in\vee q}\, \mu$, a contradiction. So $\mu(x) \geq \min(\mu(x^2), 0.5)$. Conversely let $(\mu(x) \geq \min(\mu(x^2), 0.5)$ and let $x_r{}^2 \in \mu$, for all $x \in N$. Then $\mu(x) \geq \min(\mu(x^2), 0.5) \geq \min(r, 0.5) = 0.5$ or r according as $r > 0.5$ or $r \leq 0.5$. Hence $x_r \in \vee q \mu$. Thus μ is an $(\in,\in\vee q)$-fuzzy semiprime bi-ideal of N.

Proposition 3.4. Every $(\in,\in\vee q)$-fuzzy prime bi-ideal of N is a fuzzy semiprime bi-ideal of N.

Theorem 3.5. μ be an $(\in,\in\vee q)$-fuzzy prime bi-ideal of N if and only if for all $t \in (0,0.5]$, if μ_t is non-empty and μ_t is a prime bi-ideal of N.

Proof : Let μ is an $(\in,\in\vee q)$- fuzzy prime bi-ideal of N and $t \in (0, 0.5]$ such that μ_t is
non empty. To prove μ_t is a prime bi-ideal of N. Suppose $x, y \in N$ and $x, y \in \mu_t$. Then since μ is an $(\in,\in\vee q)$- fuzzy prime bi-ideal, we have $\mu(x - y) \geq \min(\mu(x), \mu(y), 0.5) \geq \min(t, 0.5) = t$. So $(x - y) \in \mu_t$.

Now let $z \in N$. Suppose $z \in \mu_t N \mu_t \cap \mu_t N * \mu_t$. Then there exists x, y, a_1, a_2, b $\in \mu_t$ and $n_1, n_2, n_3 \in N$ such that $z = a_1 n_2 (a_2 n_3 + b) - a_1 n_2 a_2 n_3$. Thus $\mu(x) \geq t$, $\mu(y) \geq t$, $\mu(a_1) \geq t$, $\mu(a_2) \geq t$, $\mu(b) \geq t$. Now

$$(\mu N \mu \cap \mu N * \mu)(z) = \min\{(\mu N \mu)(z), (\mu N * \mu)(z)\}$$

$$= \min\{\sup_{z=xn_1y}\{\min\{\mu(x), \mu(y)\}\}, \sup_{z=a_1n_2(a_2n_{3+}+b)-a_1n_2a_2n_3} \min\{\{\mu(a_1), \mu(b)\}\}\} \geq t$$

We have

$$\min\{(\mu N \mu)(z), (\mu N * \mu)(z), 0.5\}$$

$$= \min[\min\{\sup_{z=xn_1y}\{\min\{\mu(x), \mu(y)\}\}, \sup_{z=a_1n_2(a_2n_{3+}+b)-a_1n_2a_2n_3} \min\{\{\mu(a_1), \mu(b)\}\}\}, 0.5]$$

$\geq \min(t, 0.5)$. Since μ is an $(\in, \in \vee q)$-fuzzy prime bi-ideal of N, we have $\mu(z) \geq t$. So $z \in \mu_t$. Hence μ_t is a bi-ideal of N. Again let $x, y \in N$ be such that $xy \in \mu_t$. Then $\mu(xy) \geq t$. Since μ is an $(\in, \in \vee q)$-fuzzy prime bi-ideal of N, we have $\mu(x) \vee \mu(y) \geq \min(\mu(xy), 0.5) \geq \min(t, 0.5)$. which gives $\mu(x) \geq t, \mu(y) \geq t$. Hence $x \in \mu_t$ or $y \in \mu_t$. Thus μ_t is a prime bi-ideal of N.

Conversely let μ_t is a bi-ideal of N where $t \in (0, 0.5]$. Consider

$$(\mu N \mu \cap \mu N * \mu)(z) = \min\{(\mu N \mu)(z), (\mu N * \mu)(z)\}$$

$$= \min\{\sup_{z=xn_1y}\{\min\{\mu(x), \mu(y)\}\}, \sup_{z=xn_1y=a_1n_2(a_2+b)-a_1n_2a_2} \min\{\{\mu(a_1), \mu(b)\}\}\}$$

$$= \sup_{z=xn_1y=a_1n_2(a_2+b)-a_1n_2a_2} \{\min\{\mu(x), \mu(y), \mu(a_1), \mu(b)\}\}$$

Let $\mu(x) = t_1 < \mu(y) = t_2 < \mu(a_1) = t_3 < \mu(b) = t_4$ then $\mu_{t_1} \supseteq \mu_{t_2} \supseteq \mu_{t_3} \supseteq \mu_{t_4}$. Then $x, y, a_1, b \in \mu_t$ and $z = xn_1y \in \mu_{t_1} N \mu_{t_1}$ and $z = a_1n_2(a_2 + b) - a_1n_2a_2 \in \mu_{t_1} N * \mu_{t_1}$. Thus $z \in \mu_{t_1} N \mu_{t_1} \cap \mu_{t_1} N * \mu_{t_1}$. Thus μ is an fuzzy bi-ideal of N. Also every fuzzy bi-ideal of N is an $(\in, \in \vee q)$-fuzzy bi-ideal of N (Remarks 2.18.). Hence μ is an $(\in, \in \vee q)$-fuzzy bi-ideal of N.

Again let x, y$\in$ N be such that μ(xy) = t and $(xy)_t \in \mu$. Since μ_t is a prime bi-ideal of N, we have x $\in \mu_t$ or y$\in \mu_t$. Which gives $x_t \in\vee q\mu$ or $y_t \in \vee q\mu$. Thus μ is an $(\in,\in\vee q)$-fuzzy prime bi-ideal of N.

Theorem 3.6. Let μ be an $(\in,\in\vee q)$-fuzzy bi-ideal of N. Then the following statements are equivalent,

(1) μ be an $(\in,\in\vee q)$-fuzzy prime bi-ideal of N.

(2) For any fuzzy bi-ideal λ and δ in N

$\lambda\circ\delta \subseteq \mu$ implies $\lambda \subseteq \vee q\, \mu$ or $\delta \subseteq \vee q\, \mu$

Proof: (1) implies (2). Let μ be an $(\in,\in\vee q)$-fuzzy bi-ideal of N. Let λ and δ be any fuzzy bi-ideal of N. If $\lambda \overline{\subseteq \vee q}\, \mu$ then there exists $x_t \in \lambda$ such that $x_t \overline{\in\vee q}\, \mu$. Then for all $y_r \in \delta$ we have $x_t \circ y_r \in \lambda\circ\delta \subseteq \mu$. But $x_t \overline{\in\vee q}\, \mu$. Hence $y_r \in\vee q\, \mu$. Since μ is an$(\in,\in\vee q)$-fuzzy prime bi-ideal of N, we have $\delta \subseteq \vee q\, \mu$. Hence (1) implies (2). Clearly (2) implies (1).

Theorem 3.7. Let A be a non empty subset of N. Then A is a prime bi-ideal of N if and only if χ_A is an $(\in,\in\vee q)$-fuzzy prime bi-ideal of N.

Theorem 3.8. Let μ be an $(\in,\in\vee q)$-fuzzy ideal of N. Then μ be an $(\in,\in\vee q)$-fuzzy semiprime bi-ideal of N if and only if for all $t\in(0,0.5]$, if μ_t is non-empty and μ_t is a semiprime bi-ideal of N.

Theorem 3.9. Let μ and λ be any two $(\in,\in\vee q)$-fuzzy semiprime bi-ideal of N. Then $\mu\cap\lambda$ is also an $(\in,\in\vee q)$-fuzzy semiprime bi-ideal of N.

Proof : Let μ and λ be any two $(\in,\in\vee q)$-fuzzy semiprime bi-ideal of N. Then $\mu\cap\lambda$ is an $(\in,\in\vee q)$-fuzzy bi-ideal of N (Remarks 2.19.).

Let $x \in N$. Then $(\mu \cap \lambda)(x) = \mu(x) \wedge \lambda(x)$

$$\geq \{ \min(\mu(x^2), 0.5)\} \wedge \{ \min(\lambda(x^2), 0.5)\}$$

$$\geq \min\{((\mu \cap \lambda)(x^2), 0.5)\}.$$

Thus $\mu \cap \lambda$ is also an $(\in, \in \vee q)$- fuzzy semiprime bi-ideal of N.

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