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The Application of Integral Equations for the Solution of a Satellite under the Gravitational Field of an Oblate Spheroid

Dr. Kumari Ranjana

Asst. Professor, Department of Mathematics
Lakshmibai College,
University of Delhi

ABSTRACT

The launching of artificial satellites under the various fields of forces has created a revolution in the field of space research. Many researches numerical as well as analytical have been developed to investigate the problem. Here a new method has been explored to deal with the problem .It is the method of application of Integral Equations. The motion of an artificial satellite in the gravitational field of an oblate spheroid have been studied .We have found the solution by choosing a suitable kernel .The convergency and uniqueness of the solution have also been examined .The error committed in closing the approximations at nth stage has been calculated.

Keywords: Approximation, Gravitational, Oblate , kernel.

Equation of Motion of a satellite under the Gravitational Field of an Oblate Spheroid.

The equations of motion of a small mass attracted by a spheroid are [1,2]

$$\frac{d^2x}{dt^2}=\frac{\partial U}{\partial x},\frac{d^2y}{dt^2}=\frac{\partial U}{\partial y},\frac{d^2z}{dt^2}=\frac{\partial U}{\partial z},$$

$$U=\frac{\mu}{r}+\frac{\mu K_2}{r^3}\left(\frac{3}{2}Sin^2\beta-\frac{1}{2}\right)+\frac{\mu K_4}{r^5}\left(\frac{3}{8}-\frac{15}{4}Sin^2\beta+\frac{35}{8}Sin^4\beta\right)+.......$$

where U is the potential of the earth on a unit mass. Let us omit higher order terms and take

$$U = \frac{\mu}{r}+\frac{\mu K_2}{r^3}\left(\frac{3}{2}Sin^2\beta-\frac{1}{2}\right)$$

$$=\frac{\mu}{r}-\frac{\mu K_2}{2r^3}+\frac{3}{2}\frac{\mu K_2}{r^3}Sin^2\beta$$

$$\therefore U = \frac{\mu}{r}-\frac{\mu K_2}{2r^3}+\frac{3}{2}\frac{\mu K_2}{r^3}\frac{z^2}{r^2}$$

where $z^2=r^2Sin^2\beta$

and

$$\frac{\partial U}{\partial x} = -\frac{\mu x}{r^3}+\frac{3\mu K_2 x}{2r^5}-\frac{15\mu K_2 z^2 x}{2r^7},$$

$$\frac{\partial U}{\partial y}=-\frac{\mu y}{r^3}+\frac{3\mu K_2 y}{2r^5}-\frac{15\mu K_2 z^2 y}{2r^7},$$

$$\frac{\partial U}{\partial z} = -\frac{\mu z}{r^3}+\frac{3\mu K_2 z}{2r^5}-\frac{15\mu K_2 z^3}{2r^7}+\frac{3\mu K_2 z}{r^5}$$

$$\frac{d^2\vec{r}}{dt^2}=-\frac{\mu\vec{r}}{r^3}+\frac{3\mu K_2\vec{r}}{2r^5}-\frac{15\mu K_2 z^2\vec{r}}{2r^7}+\frac{3\mu K_2\overrightarrow{rs}}{r^5}\ ,where\ \overrightarrow{rs}=(0,0,z)$$

$$=-\mu\left(\frac{\vec{r}}{r^3}-K_2\left(\frac{3\vec{r}}{2r^5}-\frac{15}{2}z^2\vec{r}+\frac{3\vec{rs}}{r^5}\right)\right)$$

.... (1)

Let us introduce a vector $\vec{r_0}(t)$ where

$$\vec{r_0}(t)=\frac{t_2-t}{t_2-t_1}\vec{r_1}+\frac{t-t_1}{t_2-t_1}\vec{r_2}$$

where $\vec{r_1}$ and $\vec{r_2}$ are the two given helio-centric positions at time t_1 and t_2.Then the vector $\vec{r}(t)-\vec{r_0}(t)$ satisfies the differential equation of the second order with the null boundary value condition which completely determine the solution.

With the help of Green's function

$$G(t,t') = \frac{(t-t_1)(t_2-t')}{(t_2-t_1)}\ (t \le t')$$

$$G(t,t') = G(t',t)(t \ge t')$$

The problem reduces to the solution of the non-linear equation

$$\vec{r}(t) = \vec{r}_0(t) + \mu \int_{t_1}^{t_2} G(t,t')\left(\frac{\vec{r}}{r^3} - K_2\left\{\frac{3}{2}\frac{\vec{r}}{r^5} - \frac{15}{2}z^2\frac{\vec{r}}{r^7} + \frac{\vec{r}_s}{r^5}\right\}\right)dt'$$

.... (2)

If the moment of t_1 be taken for the initial origin of time and the difference $t_2 - t_1$ for the unit of time ,then the time t varies in the interval (0,1).The Kernel of the integral equation takes the form

$$K(t,t') = \begin{cases} t(1-t'), t \le t' \\ t'(1-t), t \ge t' \end{cases}.$$

and the equation (2) reduces to

$$\vec{r}(t) = \vec{r_0}(t) + \mu \int_0^1 K(t,t')\left(\frac{\vec{r}}{r^3} - K_2\left\{\frac{3}{2}\frac{\vec{r}}{r^5} - \frac{15}{2}z^2\frac{\vec{r}}{r^7} + \frac{\vec{r_s}}{r^5}\right\}\right)dt'$$

.... (3)

Let

$$\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$$

.... (4)

Then equating the coefficient of $\vec{i}, \vec{j}, \vec{k}$ we get

$$x = x_0 + \mu \int_0^1 K(t,t')\left(\frac{x}{r^3} - K_2\left\{\frac{3}{2}\frac{x}{r^5} - \frac{15}{2}z^2\frac{x}{r^7}\right\}\right)dt'$$

$$y = y_0 + \mu \int_0^1 K(t,t')\left(\frac{y}{r^3} - K_2\left\{\frac{3}{2}\frac{y}{r^5} - \frac{15}{2}z^2\frac{y}{r^7}\right\}\right)dt'$$

$$z = z_0 + \mu \int_0^1 K(t,t')\left(\frac{z}{r^3} - K_2\left\{\frac{3}{2}\frac{z}{r^5} - \frac{15}{2}\frac{z^3}{r^7} + \frac{3z}{r^5}\right\}\right)dt'$$

.... (5)

which can be solved by Newton's method .For zeroth approximation we shall take

$$x_0(t) = x_0, y_0(t) = y_0, z_0(t) = z_0.$$

Consider the three integral equations

$$x(t) = x_0 + \mu \int_0^1 U_1\left(t, t', x(t'), y(t'), z(t')\right) dt'$$

$$y(t) = y_0 + \mu \int_0^1 U_2\left(t, t', x(t'), y(t'), z(t')\right) dt'$$

$$z(t) = z_0 + \mu \int_0^1 U_3\left(t, t', x(t'), y(t'), z(t')\right) dt'$$

.... (6)

where the zeroth approximation may be taken as $x_0(t), y_0(t), z_0(t)$.

2.Determination of the first approximation of the solution :

For the determination of the first approximation we shall give the increment to $x_0(t), y_0(t), z_0(t)$.

$$\Delta_1^1 = x'(t) - x_0(t)$$

$$\Delta_2^1 = y'(t) - y_0(t)$$

$$\Delta_3^1 = z'(t) - z_0(t)$$

defined by linear integral equations

$$\Delta_1^1 - \mu \int_0^1 \left\{ \left(\frac{\partial U_1}{\partial x}\right)_0 \Delta_1^1 + \left(\frac{\partial U_1}{\partial y}\right)_0 \Delta_2^1 + \left(\frac{\partial U_1}{\partial z}\right)_0 \Delta_3^1 \right\} dt'$$

$$= \mu \int_0^1 U_1\left(t, t', x(t'), y(t'), z(t')\right) dt'$$

$$\Delta_2^1 - \mu \int_0^1 \left\{ \left(\frac{\partial U_2}{\partial x}\right)_0 \Delta_1^1 + \left(\frac{\partial U_2}{\partial y}\right)_0 \Delta_2^1 + \left(\frac{\partial U_2}{\partial z}\right)_0 \Delta_3^1 \right\} dt'$$

$$= \mu \int_0^1 U_2\left(t, t', x(t'), y(t'), z(t')\right) dt'$$

$$\Delta_3^1 - \mu \int_0^1 \left\{ \left(\frac{\partial U_3}{\partial x}\right)_0 \Delta_1^1 + \left(\frac{\partial U_3}{\partial y}\right)_0 \Delta_2^1 + \left(\frac{\partial U_3}{\partial z}\right)_0 \Delta_3^1 \right\} dt'$$

$$= \mu \int_0^1 U_3\left(t, t', x(t'), y(t'), z(t')\right) dt'$$

.... (7)

The second approximation $x^2(t), y^2(t), z^2(t)$ can be obtained if we can obtain $\Delta_1^2, \Delta_2^2, \Delta_3^2$ from the following:-

$$\Delta_1^2 - \mu\int_0^1 \left\{\left(\frac{\partial U_1}{\partial x}\right)_1 \Delta_1^2 + \left(\frac{\partial U_1}{\partial y}\right)_1 \Delta_2^2 + \left(\frac{\partial U_1}{\partial z}\right)_1 \Delta_3^2\right\} dt'$$

$$= \mu\int_0^1 U_1\left(t,t',x'(t'),y'(t'),z'(t')\right)dt' + x^0(t') - x'(t)$$

$$\Delta_2^2 - \mu\int_0^1 \left\{\left(\frac{\partial U_2}{\partial x}\right)_1 \Delta_1^2 + \left(\frac{\partial U_2}{\partial y}\right)_1 \Delta_2^2 + \left(\frac{\partial U_2}{\partial z}\right)_1 \Delta_3^2\right\} dt'$$

$$= y^0(t') - y'(t') + \mu\int_0^1 U_2\left(t,t',x'(t'),y'(t'),z'(t')\right)dt'$$

$$\Delta_3^2 - \mu\int_0^1 \left\{\left(\frac{\partial U_3}{\partial x}\right)_1 \Delta_1^2 + \left(\frac{\partial U_3}{\partial y}\right)_1 \Delta_2^2 + \left(\frac{\partial U_3}{\partial z}\right)_1 \Delta_3^2\right\} dt'$$

$$= z^0(t') - z'(t') + \mu\int_0^1 U_3\left(t,t',x'(t'),y'(t'),z'(t')\right)dt'$$

where

$$\Delta_1^2 = x^2(t) - x'(t)$$

$$\Delta_2^2 = y^2(t) - y'(t)$$

$$\Delta_3^2 = z^2(t) - z'(t)$$

Lastly the n $_{th}$ approximation $x^n(t), y^n(t), z^n(t)$ can be obtained, if we can obtain $\Delta_1^n, \Delta_2^n, \Delta_3^n$ from the following integral equations

$$\Delta_1^n - \mu\int_0^1 \left\{\left(\frac{\partial U_1}{\partial x}\right)_{n-1} \Delta_1^n + \left(\frac{\partial U_1}{\partial y}\right)_{n-1} \Delta_2^n + \left(\frac{\partial U_1}{\partial z}\right)_{n-1} \Delta_3^n\right\} dt'$$

$$= x^0(t) - x^{n-1}(t) + \mu\int_0^1 U_1\left(t,t',x^{n-1}(t'),y^{n-1}(t'),z^{n-1}(t')\right)dt'$$

$$\Delta_2^n - \mu\int_0^1 \left\{\left(\frac{\partial U_2}{\partial x}\right)_{n-1} \Delta_1^n + \left(\frac{\partial U_2}{\partial y}\right)_{n-1} \Delta_2^n + \left(\frac{\partial U_2}{\partial z}\right)_{n-1} \Delta_3^n\right\} dt'$$

$$= y^0(t) - y^{n-1}(t) + \mu\int_0^1 U_2\left(t,t',x^{n-1}(t'),y^{n-1}(t'),z^{n-1}(t')\right)dt'$$

$$\Delta_3^n - \mu\int_0^1 \left\{\left(\frac{\partial U_3}{\partial x}\right)_{n-1}\Delta_1^n + \left(\frac{\partial U_3}{\partial y}\right)_{n-1}\Delta_2^n + \left(\frac{\partial U_3}{\partial z}\right)_{n-1}\Delta_3^n\right\}dt'$$

$$= z^0(t) - z^{n-1}(t) + \mu\int_0^1 U_3\left(t,t',x^{n-1}(t'),y^{n-1}(t'),z^{n-1}(t')\right)dt'$$

.... (8)

Now in our case $U_i, \frac{\partial U_i}{\partial x}, \frac{\partial U_i}{\partial y}, \frac{\partial U_i}{\partial z}$ have the following form

$$U_1 = K(t,t')\left(\frac{x}{r^3} - K_2\left\{\frac{3x}{2r^5} - \frac{15z^2x}{2r^7}\right\}\right)$$

$$U_2 = K(t,t')\left(\frac{y}{r^3} - K_2\left\{\frac{3y}{2r^5} - \frac{15z^2y}{2r^7}\right\}\right)$$

$$U_3 = K(t,t')\left(\frac{z}{r^3} - K_2\left\{\frac{3z}{2r^5} - \frac{15z^3}{2r^7} + \frac{3z}{r^5}\right\}\right)$$

$$\frac{\partial U_1}{\partial x} = \frac{K(t,t')}{r^3}\left(1 - \frac{3x^2}{r^2}\right) - \frac{3K(t,t')}{2\,r^5}K_2\left(1 - \frac{5x^2}{r^2}\right) + \frac{15}{2}K_2\frac{K(t,t')}{r^7}\left(1 - \frac{7x^2}{r^2}\right)z^2$$

$$\frac{\partial U_1}{\partial y} = -\frac{K(t,t')}{r^5}3xy + K(t,t')K_2\frac{15xy}{2r^7} - K(t,t')K_2\frac{105z^2xy}{2r^9}$$

$$\frac{\partial U_1}{\partial z} = -\frac{K(t,t')}{r^5}3xy + K(t,t')K_2\frac{15xz}{2r^7} - K(t,t')K_2\frac{15x}{2r^7}\left(2z - \frac{7z^3}{r^2}\right)$$

$$\frac{\partial U_2}{\partial x} = \frac{K(t,t')}{r^5}\,3xy + K(t,t')\,K_2\frac{15y}{r^7} - K(t,t')K_2\frac{105z^2yx}{2r^9}$$

$$\frac{\partial U_2}{\partial y} = \frac{K(t,t')}{r^3}\left(1 - \frac{3y^2}{r^2}\right) - \frac{3K(t,t')}{2\,r^5}K_2\left(1 - \frac{5y^2}{r^2}\right) + \frac{15}{2}K_2\frac{K(t,t')}{r^7}\left(1 - \frac{7y^2}{r^2}\right)z^2$$

$$\frac{\partial U_2}{\partial z} = -\frac{K(t,t')}{r^5}3yz + \frac{K(t,t')}{2r^7}K_2 15yz - K(t,t')K_2\frac{15y}{2r^7}\left(2z - \frac{7z^3}{r^2}\right)$$

$$\frac{\partial U_3}{\partial x} = -\frac{K(t,t')}{r^5}3yz + \frac{K(t,t')}{2r^7}K_2 15xz - \frac{7K(t,t')}{2r^9}\frac{15z^2x}{2} + \frac{15K(t,t')}{r^7}K_2 xz$$

$$\frac{\partial U_3}{\partial y} = -\frac{K(t,t')}{r^5}3yz + \frac{K(t,t')}{2r^7}15yz - \frac{75K(t,t')K_2 yz^2}{2r^9} + \frac{15K(t,t')yz}{r^7}$$

$$\frac{\partial U_3}{\partial z} = -\frac{K(t,t')}{r^3}\left(1 - \frac{3z^2}{r^2}\right) - \frac{3K(t,t')K_2}{2r^5}\left(1 - \frac{5z^2}{r^2}\right) + \frac{15K(t,t')K_2}{2r^7}\left(3z^2 - \frac{7z^4}{r^2}\right) - \frac{3K(t,t')}{r^5}K_2\left(1 - \frac{5z^2}{r^2}\right)$$

.... (9)

$$\left|\int_0^1 U_i(t,t')dt'\right| \le \frac{M^2}{8}\frac{\max\limits_{0\le t\le 1}\{x(t),y(t),z(t)\}}{\min\limits_{0\le t\le 1}\{r^3,r^5\}}$$

$$\left|\int_0^1 \frac{\partial U_i}{\partial x}dt'\right|,\left|\int_0^1 \frac{\partial U_i}{\partial y}dt'\right| \text{ and } \left|\int_0^1 \frac{\partial U_i}{\partial z}dt'\right| \text{ are } \le \frac{M^2}{8}\frac{\max\limits_{0\le t\le 1}\{x^2(t),y^2(t),z^2(t)\}}{\min\limits_{0\le t\le 1}\{r^5,r^7\}}$$

.... (10)

where , $M^2 = 63, \int_1^0 K(t,t')dt' \le 1$ and $\left|\frac{z^2}{r^2}\right| = \left|\sin^2\beta\right| \le 1.$

3 .Convergence of the successive approximation:

Let us now find out the conditions under which the process of successive approximation given by (7) converge. Let us write them down in the following form.

$$x^n(t) = x^0(t) + \mu\int_0^1 U_1\left(t,t',x^{n-1}(t'),y^{n-1}(t'),z^{n-1}(t')\right)dt'$$

$$+\mu\int_0^1\left\{\left(\frac{\partial U_1}{\partial x}\right)_{n-1}\Delta_1^n + \left(\frac{\partial U_1}{\partial y}\right)_{n-1}\Delta_2^n + \left(\frac{\partial U_1}{\partial z}\right)_{n-1}\Delta_3^n\right\}dt'$$

$$y^n(t) = y^0(t) + \mu\int_0^1 U_2\left(t,t',x^{n-1}(t'),y^{n-1}(t'),z^{n-1}(t')\right)dt'$$

$$+\mu\int_0^1\left\{\left(\frac{\partial U_2}{\partial x}\right)_{n-1}\Delta_1^n + \left(\frac{\partial U_2}{\partial y}\right)_{n-1}\Delta_2^n + \left(\frac{\partial U_2}{\partial z}\right)_{n-1}\Delta_3^n\right\}dt'$$

$$z^n(t) = z^0(t) + \mu\int_0^1 U_3\left(t,t',x^{n-1}(t'),y^{n-1}(t'),z^{n-1}(t')\right)dt'$$

$$+\mu\int_0^1\left\{\left(\frac{\partial U_3}{\partial x}\right)_{n-1}\Delta_1^n+\left(\frac{\partial U_3}{\partial y}\right)_{n-1}\Delta_2^n+\left(\frac{\partial U_3}{\partial z}\right)_{n-1}\Delta_3^n\right\}dt'$$

.... (11)

Subtracting $x^{n-1}(t)$ from $x^n(t)$, $y^{n-1}(t)$ from $y^n(t)$ and $z^{n-1}(t)$ from $z^n(t)$, we get

$$\Delta_1^n = \mu\int_0^1\left[U_1\left(t,t',x^{n-1}(t'),y^{n-1}(t'),z^{n-1}(t')\right)-U_1\left(t,t',x^{n-2}(t'),y^{n-2}(t'),z^{n-2}(t')\right)\right]dt'$$

$$+\mu\int_0^1\left\{\left(\frac{\partial U_1}{\partial x}\right)_{n-1}\Delta_1^n+\left(\frac{\partial U_1}{\partial y}\right)_{n-1}\Delta_2^n+\left(\frac{\partial U_1}{\partial z}\right)_{n-1}\Delta_3^n\right\}dt' - \mu\int_0^1\left\{\left(\frac{\partial U_1}{\partial x}\right)_{n-2}\Delta_1^{n-1}+\left(\frac{\partial U_1}{\partial y}\right)_{n-2}\Delta_2^{n-1}+\left(\frac{\partial U_1}{\partial z}\right)_{n-2}\Delta_3^{n-1}\right\}dt'$$

$$\Delta_2^n = \mu\int_0^1\left[U_2\left(t,t',x^{n-1}(t'),y^{n-1}(t'),z^{n-1}(t')\right)-U_2\left(t,t',x^{n-2}(t'),y^{n-2}(t'),z^{n-2}(t')\right)\right]dt'$$

$$+\mu\int_0^1\left\{\left(\frac{\partial U_2}{\partial x}\right)_{n-1}\Delta_1^n+\left(\frac{\partial U_2}{\partial y}\right)_{n-1}\Delta_2^n+\left(\frac{\partial U_2}{\partial z}\right)_{n-1}\Delta_3^n\right\}dt' - \mu\int_0^1\left\{\left(\frac{\partial U_2}{\partial x}\right)_{n-2}\Delta_1^{n-1}+\left(\frac{\partial U_2}{\partial y}\right)_{n-2}\Delta_2^{n-1}+\left(\frac{\partial U_2}{\partial z}\right)_{n-2}\Delta_3^{n-1}\right\}dt'$$

$$\Delta_3^n = \mu\int_0^1\left[U_3\left(t,t',x^{n-1}(t'),y^{n-1}(t'),z^{n-1}(t')\right)-U_3\left(t,t',x^{n-2}(t'),y^{n-2}(t'),z^{n-2}(t')\right)\right]dt'$$

$$+\mu\int_0^1\left\{\left(\frac{\partial U_3}{\partial x}\right)_{n-1}\Delta_1^n+\left(\frac{\partial U_3}{\partial y}\right)_{n-1}\Delta_2^n+\left(\frac{\partial U_3}{\partial z}\right)_{n-1}\Delta_3^n\right\}dt' - \mu\int_0^1\left\{\left(\frac{\partial U_3}{\partial x}\right)_{n-2}\Delta_1^{n-1}+\left(\frac{\partial U_3}{\partial y}\right)_{n-2}\Delta_2^{n-1}+\left(\frac{\partial U_3}{\partial z}\right)_{n-2}\Delta_3^{n-1}\right\}dt'$$

Denoting by $\Delta^n = \max_{0\le t\le 1}\left\{\left|\Delta_1^n\right|,\left|\Delta_2^n\right|,\left|\Delta_3^n\right|\right\}$

$$\Phi^n = \max_{0\le t\le 1}\left\{x^n, y^n, z^n\right\}$$

and writing the difference $U_i\left(t,t',x^{n-1}(t'),y^{n-1}(t'),z^{n-1}(t')\right)-U_i\left(t,t',x^{n-2}(t'),y^{n-2}(t'),z^{n-2}(t')\right)$

In the Lagrangian form $\left(\frac{\partial U_1}{\partial x}\right)\Delta_1^{n-1}+\left(\frac{\partial U_1}{\partial y}\right)\Delta_2^{n-1}+\left(\frac{\partial U_1}{\partial y}\right)\Delta_3^{n-1}$

where the derivatives $\frac{\partial U_1}{\partial x},\frac{\partial U_1}{\partial y},\frac{\partial U_1}{\partial z}$ are taken at some intermediate point

$$x = x^{n-1} + \theta_1\left(x^{n-1}-x^{n-2}\right)$$

$$y = y^{n-1} + \theta_2\left(y^{n-1}-y^{n-2}\right)$$

$$z = z^{n-1} + \theta_3\left(z^{n-1}-z^{n-2}\right)$$

where $|\theta_i| \le 1, i=1,2,3$

We shall have

$$\Delta^n \le \frac{\frac{3\mu M^2}{8}\max\limits_{0\le t\le 1}\{(\Phi^{n-2})^2,(\Phi^{n-1})^2\}\Delta^{n-1}}{\min\limits_{0\le t\le 1}(r_{n-1}^5,r_{n-2}^5,r_{n-1}^7,r_{n-2}^7)} + \frac{\frac{3\mu M^2}{8}\max\limits_{0\le t\le 1}\{(\Phi^{n-1})^2\}\Delta^n}{\min\limits_{0\le t\le 1}(r_{n-1}^5,r_{n-1}^7)}$$

$$+ \frac{\frac{3\mu M^2}{8}\max\limits_{0\le t\le 1}\{(\Phi^{n-2})^2\}\Delta^{n-1}}{\min\limits_{0\le t\le 1}(r_{n-2}^5,r_{n-2}^7)}$$

or,

$$\Delta^n \le \frac{\frac{\frac{3\mu M^2}{8}\max\limits_{0\le t\le 1}\{(\Phi^{n-2})^2,(\Phi^{n-1})^2\}}{\min\limits_{0\le t\le 1}(r_{n-1}^5,r_{n-1}^7 r_{n-2}^5,r_{n-2}^7)} + \frac{\frac{3\mu M^2}{8}\max\limits_{0\le t\le 1}\{(\Phi^{n-2})^2\}\Delta^n}{\min\limits_{0\le t\le 1}(r_{n-2}^5,r_{n-2}^7)}}{1-\frac{\frac{3\mu M^2}{8}\max\limits_{0\le t\le 1}\{(\Phi^{n-1})^2\}}{\min\limits_{0\le t\le 1}(r_{n-1}^5,r_{n-1}^7)}}\Delta^{n-1}$$

$$\Delta^n \le \frac{\frac{\frac{3\mu M^2}{4}\max\limits_{0\le t\le 1}\{(\Phi^{n-1})^2,(\Phi^{n-2})^2\}}{\min\limits_{0\le t\le 1}(r_{n-1}^5,r_{n-1}^7,r_{n-2}^5,r_{n-2}^7)}}{1-\frac{\frac{3\mu M^2}{8}\max\limits_{0\le t\le 1}(\Phi^{n-1})^2}{\min\limits_{0\le t\le 1}(r_{n-1}^5,r_{n-1}^7)}}\Delta^{n-1}$$

or,

$$\Delta^n \le \left(\frac{\frac{\frac{3\mu M^2}{4}\max\limits_{0\le t\le 1}\{(\Phi^{\kappa})^2\}}{\min\limits_{0\le t\le 1}(r_k^5,r_k^7)}}{1-\frac{\frac{3\mu M^2}{8}\max\limits_{0\le t\le 1}(\Phi^k)^2}{\min\limits_{0\le t\le 1}(r_k^5,r_k^7)}}\right)\Delta^1$$

.... (12)

under the condition

$$\frac{3\mu M^2}{8}\frac{\max\limits_{0\le t\le 1}(\Phi^K)^2}{\min\limits_{0\le t\le 1}(r_K^5,r_K^7)}<1$$

.... (13)

Let us now prove by the method of Mathematical induction that all $r_n \geq Kh$ and $\Phi^n \leq K' \Phi^0$ where K is some positive number < 1 and $K' > 1$ and h is the length of the perpendicular drawn from the origin on the chord joining $\vec{r}_1$ and $\bar{r}_2$. For $n=0,$ the validity of the inequality is obvious as $r_0 > h.$ Let us assume that they are true for all r_j and $\Phi^j, j=1,2,3,....(n-1)$.

i.e ,

$r_j \geq Kh$ and $\Phi^j \leq K' \Phi^0$ .... (14)

It can be shown that the inequalities hold for $j=n$

and as the consequence of (14) we can say that the inequalities hold for an arbitrary n

By Mathematical induction under condition

$$\frac{3\mu M^2}{8} \frac{K_1^2 \Phi^{0^2}}{K_1 h_1} < 1 \qquad .... (15)$$

where $K_1 h_1$ is minimum of $K^7 h^7, K^5 h^5, K^3 h^3$

We have

$$\Delta^n \leq \left(\frac{\frac{3\mu M^2}{4} \frac{K_1^2 \Phi^{0^2}}{K_1 h_1}}{1 - \frac{3\mu M^2}{8} \frac{K_1^2 \Phi^{0^2}}{K_1 h_1}} \right)^{n-1} \Delta^1 \qquad .... (16)$$

where,

$$\Delta^1 \leq \frac{\frac{1}{8} \frac{\mu M^2 \Phi^0}{\min\limits_{0 \leq t \leq 1} (r_0^5, r_0^3)}}{1 - \frac{3}{8} \frac{\mu M^2 \Phi^{0^2}}{\min\limits_{0 \leq t \leq 1} (r_0^7, r_0^5)}} \qquad .... (17)$$

r_0 and Φ^0 satisfy the condition $\frac{3}{8} \frac{\mu M^2 \Phi^{0^2}}{\min\limits_{0 \leq t \leq 1} (r_0^7, r_0^5)} < 1$

We have from the relation (11)

$$\left|\Phi^n-\Phi^0\right|\le\frac{\mu M^2}{8}\frac{\max\limits_{0\le t\le 1}\Phi^{n-1}}{\min\limits_{0\le t\le 1}(r_{n-1}^5,r_{n-1}^3)}+\frac{3\mu M^2}{8}\frac{\max\limits_{0\le t\le 1}\left(\Phi^{n-1}\right)^2}{\min\limits_{0\le t\le 1}(r_{n-1}^7,r_{n-1}^5)}\Delta^n$$

or ,

$$\left|\Phi^n-\Phi^0\right|\le\frac{\mu M^2}{8}\frac{K^1\Phi^0}{K_1h_1}+\frac{3\mu M}{8}\frac{K^{1^2}\Phi^{0^2}}{K_1h_1}\left(\frac{\frac{3\mu M^2}{4}\frac{K^{1^2}\Phi^{0^2}}{K_1h_1}}{1-\frac{3\mu M^2}{8}\frac{K^{1^2}\Phi^{0^2}}{K_1h_1}}\right)^{n-1}\left(\frac{\frac{1}{8}\frac{\mu M^2\Phi^0}{\min\limits_{0\le t\le 1}(r_0^5,r_0^3)}}{1-\frac{3}{8}\frac{\mu M^2\Phi^{0^2}}{\min\limits_{0\le t\le 1}(r_0^7,r_0^5)}}\right)$$

Also we have

$$\frac{\frac{3}{4}\frac{\mu M^2K'^2\Phi^{0^2}}{K_1h_1}}{1-\frac{3}{8}\frac{\mu M^2K'^2\Phi^{0^2}}{K_1h_1}}<1$$

or ,

$$\frac{3}{4}\frac{\mu M^2K'^2\Phi^{0^2}}{K_1h_1}+\frac{3}{8}\frac{\mu M^2K'^2\Phi^{0^2}}{K_1h_1}<1$$

or,

$$\frac{9}{8}\frac{\mu M^2K'^2\Phi^{0^2}}{K_1h_1}<1$$

.... (18)

Then we have ,

$$\left|\Phi^n-\Phi^0\right|\le\frac{1}{8}\frac{\mu M^2K'\Phi^0}{K_1h_1}\left(1+\tfrac{1}{3}\frac{1}{1-\frac{3}{8}\frac{\mu M^2\Phi^{0^2}}{\min\limits_{0\le t\le 1}(r_0^7,r_0^5)}}\right)$$

or,

$$\Phi^n\le\Phi^0+\frac{1}{8}\frac{\mu M^2K'\Phi^0}{K_1h_1}\left(1+\tfrac{1}{3}\frac{1}{1-\frac{3}{8}\frac{\mu M^2\Phi^{0^2}}{\min\limits_{0\le t\le 1}(r_0^7,r_0^5)}}\right)$$

Let us now try to find out the value of r_n

$$\vec{r_n}=x^n\vec{i}+y^n\vec{j}+z^n\vec{k}$$

$$\vec{r_n}=(x^n-x^0)\vec{i}+(y^n-y^0)\vec{j}+(z^n-z^0)\vec{k}+(x^0\vec{i}+y^0\vec{j}+z^0\vec{k})$$

or, $\left|\vec{r_n}-\vec{r_0}\right| \le \left|\Phi^n - \Phi^0\right| \left|\vec{i}+\vec{j}+\vec{k}\right|$

or, $\left|\vec{r_n}-\vec{r_0}\right| \le \sqrt{3}\left|\Phi^n - \Phi^0\right|$

or, $$\left|\vec{r_n}-\vec{r_0}\right| \le \sqrt{3}\frac{\mu M^2 K' \Phi^0}{K_1 h_1}\left(1+\frac{1}{3}\frac{1}{1-\frac{3}{8}\frac{\mu M^2 {\Phi^0}^2}{\min\limits_{0\le t\le 1}(r_0^7, r_0^5)}}\right)$$

or, $$\left|\vec{r_n}\right| \ge h - \sqrt{3}\frac{\mu M^2 K' \Phi^0}{K_1 h_1}\left(1+\frac{1}{3}\frac{1}{1-\frac{3}{8}\frac{\mu M^2 {\Phi^0}^2}{\min\limits_{0\le t\le 1}(r_0^7, r_0^5)}}\right)$$

Let us now define K and K' such that

$$\Phi^0 + \frac{1}{8}\frac{\mu M^2 K' \Phi^0}{K_1 h_1}\left(1+\frac{1}{3}\frac{1}{1-\frac{3}{8}\frac{\mu M^2 {\Phi^0}^2}{\min\limits_{0\le t\le 1}(r_0^7, r_0^5)}}\right) \le K' \Phi^0$$

.... (19)

$$h - \sqrt{3}\frac{\mu M^2 K' \Phi^0}{K_1 h_1}\left(1+\frac{1}{3}\frac{1}{1-\frac{3}{8}\frac{\mu M^2 {\Phi^0}^2}{\min\limits_{0\le t\le 1}(r_0^7, r_0^5)}}\right) \ge Kh$$

.... (20)

Then we have $\Phi^n \le K' \Phi^0$ and $r_n \ge Kh$.Which proves the inequality for arbitrary n. (19) and (20) can be put in the form

$$K' \ge \frac{1}{1-\frac{1}{8}\frac{\mu M^2 K' \Phi^0}{K_1 h_1}\left(1-\frac{3}{8}\frac{\mu M^2 {\Phi^0}^2}{\min\limits_{0\le t\le 1}(r_0^7, r_0^5)}\right)}$$

.... (21)

$$K \le 1-\sqrt{3}\frac{\mu M^2 K' \Phi^0}{K_1 h_1 h}\left(1+\frac{1}{3}\frac{1}{1-\frac{3}{8}\frac{\mu M^2 \Phi^{0^2}}{\min\limits_{0\le t\le 1}(r_0^7, r_0^5)}}\right)$$

.... (22)

If K and K' can be found such that (18) ,(21) and (22) are satisfied then the successive approximation given by (9) converges. From (21) and (22) putting the equality sign K and K' can be found by the method of successive approximation, by taking $K=K'=1$ for the first approximation .Under this for small value of μ this process must converge ,firstly as in the right hand side there are expressions very near to unity.

The sequence of functions $(x^n),(y^n),(z^n)$ under the fulfillment of the condition (18) by virtue of inequality (16) converges to the function x,y and z, which are the solution of the system (6) .This can be verified by taking $n\to\infty$ in equation(8).The law of passage to the limit under the sign of integration follows from the uniform convergence of the integrand , as values (16) do not depend upon time .We shall prove the uniqueness of the solution in the region

$$\frac{9}{8}\frac{\mu M^2 K'^2 \Phi^{0^2}}{K_1 h_1} < 1$$

Let us suppose that other solutions x', y', z' also satisfy the condition (18) .Consider the difference

$$[x'-x]=\mu\int_0^1\left[U_1(t,t',x'(t'),y'(t'),z'(t'))-U_1(t,t',x(t'),y(t'),z(t'))\right]dt$$

$$[y'-y]=\mu\int_0^1\left[U_2(t,t',x'(t'),y'(t'),z'(t'))-U_2(t,t',x(t'),y(t'),z(t'))\right]dt$$

$$[z'-z]=\mu\int_0^1\left[U_3(t,t',x'(t'),y'(t'),z'(t'))-U_3(t,t',x(t'),y(t'),z(t'))\right]dt$$

Changing increment of the function from

$$U_i(t,t',x'(t'),y'(t'),z'(t'))-U_i(t,t',x(t'),y(t'),z(t'))$$

To the differential at the point (x'', y'', z'')

$$x''=x+\theta_1(x'-x)$$

$$y''=y+\theta_2(y'-y)$$

$$z''=z+\theta_3(z'-z)$$

where $|\theta_i|\le 1, i=1,2,3....$

and using the inequality

$$|\Phi'-\Phi| = \max_{0\le t\le 1}\left(|x'-x|,|y'-y|,|z'-z|\right)$$

we can write $$|\Phi'-\Phi| \le \frac{3\mu M^2}{8}\frac{\max\limits_{0\le t\le 1}\Phi^2}{\min\limits_{0\le t\le 1}\left(r_0^{\,7},r_0^{\,5}\right)}|\Phi'-\Phi|$$

or , $$|\Phi'-\Phi|\left(1-\frac{3\mu M^2}{8}\frac{\max\limits_{0\le t\le 1}\Phi^2}{\min\limits_{0\le t\le 1}\left(r_0^{\,7},r_0^{\,5}\right)}\right) \le |\Phi'-\Phi|$$

The expression under the bracket is positive, hence we have $\Phi'-\Phi$. In order to get the amount of error committed by closing our calculation at the nth approximation we shall consider the difference.

$$\left|\Phi-\Phi^n\right| \le \max_{0\le t\le 1}\left(\left|x-x^n\right|,\left|y-y^n\right|,\left|z-z^n\right|\right)$$

$$\le \sum_{k=n}^{\infty}\Delta^k = \frac{\alpha^{n-1}}{1-\alpha}\Delta^1$$

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